

CRITERION OF THE BOUNDEDNESS OF SINGULAR INTEGRALS ON SPACES OF HOMOGENEOUS TYPE

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ABSTRACT. It was well known that geometric considerations enter in a decisive way in many questions of harmonic analysis. The main purpose of this paper is to provide the criterion of the boundedness for singular integrals on the Hardy spaces and as well as on its dual, particularly on BMO for spaces of homogeneous type (X, d, μ) in the sense of Coifman and Weiss, where the quasi-metric d may have no regularity and the measure μ satisfies only the doubling property. We make no additional geometric assumptions on the quasi-metric or the doubling measure and thus, the results of this paper extend to the full generality of all related previous ones, in which the extra geometric assumptions were made on both the quasi-metric d and the measure μ . To achieve our goal, we prove that the atomic Hardy spaces introduced by Coifman and Weiss coincide with the Hardy spaces defined in terms of wavelet coefficients and develop the molecule theory for this general setting. The main tools used in this paper are atomic decomposition, the orthonormal wavelet basis constructed recently by Auscher and Hytönen, the discrete Calderón-type reproducing formula, the almost orthogonal estimates, implement various stopping time arguments and the duality of the Hardy spaces with the Carleson measure spaces.

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1. INTRODUCTION

The classical theory of Calderón–Zygmund singular integral operators as well as the theory of function spaces were based on extensive use of convolution operators and on the Fourier transform. However, it is now possible to extend most of those ideas and results to spaces of homogeneous type. As Meyer remarked in his preface to [DH], “*One is amazed by the dramatic changes that occurred in analysis during the twentieth century. In the 1930s complex methods and Fourier series played a seminal role. After many improvements, mostly achieved by the Calderón–Zygmund school, the action takes place today on spaces of homogeneous type. No group structure is available, the Fourier transform is missing, but a version of harmonic analysis is still present. Indeed the geometry is conducting the analysis.*”

Spaces of homogeneous type were introduced by Coifman and Weiss in the early 1970s, in [CW1]. We say that (X, d, μ) is a *space of homogeneous type* in the sense of Coifman and Weiss if d is a quasi-metric on X and μ is a nonzero measure satisfying the doubling condition. A *quasi-metric* d on a set X is a function $d : X \times X \rightarrow [0, \infty)$ satisfying (i) $d(x, y) = d(y, x) \geq 0$ for all $x, y \in X$; (ii) $d(x, y) = 0$ if and only if $x = y$; and (iii) the *quasi-triangle inequality*: there is a constant $A_0 \in [1, \infty)$ such that for all $x, y, z \in X$,

$$(1.1) \quad d(x, y) \leq A_0[d(x, z) + d(z, y)].$$

We define the quasi-metric ball by $B(x, r) := \{y \in X : d(x, y) < r\}$ for $x \in X$ and $r > 0$. Note that the quasi-metric, in contrast to a metric, may not be Hölder regular and quasi-metric balls may not be open. We say that a nonzero measure μ satisfies the *doubling condition* if there is a constant C_μ such that for all $x \in X$ and $r > 0$,

$$(1.2) \quad \mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)) < \infty.$$

We point out that the doubling condition (1.2) implies that there exist positive constants ω (the *upper dimension* of μ) and C_ω such that for all $x \in X$, $\lambda \geq 1$ and $r > 0$,

$$(1.3) \quad \mu(B(x, \lambda r)) \leq C_\omega \lambda^\omega \mu(B(x, r)).$$

Spaces of homogeneous type include many special spaces in analysis and have many applications in the theory of singular integrals and function spaces; See [Chr, CW2, NS1, NS2] for more details. Coifman and Weiss in [CW2] introduced the atomic Hardy space $H_{cw}^p(X)$ on (X, d, μ) . To recall the atomic Hardy space, one first needs the Campanato spaces $\mathcal{C}_\alpha(X)$, $\alpha \geq 0$, consisting of those functions for which

$$(1.4) \quad \left\{ \frac{1}{\mu(B)} \int_B |f(x) - f_B|^2 d\mu(x) \right\}^{\frac{1}{2}} \leq C[\mu(B)]^\alpha,$$

where B are any quasi-metric balls, $f_B = \frac{1}{\mu(B)} \int_B f(x) d\mu(x)$ and C is independent of B . Let $\|f\|_{\mathcal{C}_\alpha(X)}$ be the infimum of all C for which (1.4) holds. The atomic Hardy space $H_{cw}^p(X)$ introduced by Coifman and Weiss is defined to be the subspace of the dual of $\mathcal{C}_\alpha(X)$, where

$\alpha = \frac{1}{p} - 1, 0 < p \leq 1$, consisting of those linear functional admitting an *atomic decomposition*

$$(1.5) \quad f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where the a_j 's are $(p, 2)$ -atoms, $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$ and the series in (1.5) converges in the dual of $\mathcal{C}_\alpha(X)$, where $\alpha = \frac{1}{p} - 1, 0 < p \leq 1$.

Here a function $a(x)$ is an $(p, 2)$ -atom if

- (i) the support of $a(x)$ is contained in a ball $B(x_0, r)$ for $r > 0$ and $x_0 \in X$;
- (ii) $\|a(x)\|_{L^2(X)} \leq \{\mu(B(x_0, r))\}^{\frac{1}{2} - \frac{1}{p}}$;
- (iii) $\int_X a(x) d\mu(x) = 0$.

The quasi-norm of f in $H_{cw}^p(X)$ is defined by $\|f\|_{H_{cw}^p(X)} = \inf\{\{\sum_{j=1}^{\infty} |\lambda_j|^p\}^{\frac{1}{p}}\}$, where the infimum is taken over all such atomic representations of f .

The atomic Hardy spaces have many applications. For example, if an operator T is bounded on $L^2(X)$ and from $H_{cw}^p(X)$ to $L^p(X)$ for some $p \leq 1$, then T is bounded on $L^q(X)$ for $1 < q \leq 2$. See [CW2] for more applications.

Even though spaces of homogeneous have many applications, however, for some applications, additional geometric assumptions were required on the quasi-metric d and the measure μ . This is because, as mentioned, the original quasi-metric d may have no regularity and quasi-metric balls, even Borel sets, may not be open. For instance, to establish the maximal function characterization for the Hardy space on spaces of homogeneous type, Macías and Segovia in [MS1] replaced the quasi-metric d by another quasi-metric d' on X such that the topologies induced on X by d and d' coincide, and d' has the following regularity property:

$$(1.6) \quad |d'(x, y) - d'(x', y)| \leq C_0 d'(x, x')^\theta [d'(x, y) + d'(x', y)]^{1-\theta}$$

for some constant C_0 , some regularity exponent $\theta \in (0, 1)$, and for all $x, x', y \in X$. Moreover, if quasi-metric balls are defined by this new quasi-metric d' , that is, $B'(x, r) := \{y \in X : d'(x, y) < r\}$ for $r > 0$, then the measure μ satisfies the following property:

$$(1.7) \quad \mu(B'(x, r)) \sim r.$$

Note that property (1.7) is much stronger than the doubling condition. Macías and Segovia [MS1] first introduced test function and distribution spaces based on the conditions (1.6) and (1.7), and then established the maximal function characterization for Hardy spaces $H_{max}^p(X)$ with $(1 + \theta)^{-1} < p \leq 1$, on spaces of homogeneous type (X, d', μ) that satisfy the regularity condition (1.6) on the quasi-metric d' and property (1.7) on the measure μ . The most remarkable work on (X, d', μ) is the Tb theorem of David, Journé and Semmes [DJS]. See also [DH] and [HS] for the Littlewood–Paley square function characterization of the Hardy, Besove and Triebel–Lizorkin spaces on such spaces (X, d', μ) .

This theme has now been developed systematically by a number of people. In [NS1], Nagel and Stein developed the product L^p ($1 < p < \infty$) theory in the setting of the

Carnot–Carathéodory spaces formed by vector fields satisfying Hörmander’s finite rank condition. The Carnot–Carathéodory spaces studied in [NS1] are spaces of homogeneous type with a smooth quasi-metric d and a measure μ satisfying the conditions $\mu(B(x, sr)) \sim s^{m+2}\mu(B(x, r))$ for $s \geq 1$ and $\mu(B(x, sr)) \sim s^4\mu(B(x, r))$ for $s \leq 1$. These conditions on the measure are weaker than property in (1.7) but are still stronger than the original doubling condition. In [HMY], motivated by the work of Nagel and Stein, Hardy spaces, namely the atomic and the Littlewood–Paley square function characterizations, were developed on spaces of homogeneous type (X, d, μ) with the quasi-metric d satisfies the regular property in (1.6) and the measure μ satisfies the above conditions which are stronger than the doubling property in (1.2).

More recently, Auscher and Hytönen [AH] constructed an orthonormal wavelet basis with Hölder regularity and exponential decay for spaces of homogeneous type in the sense of Coifman and Weiss. This result is remarkable since there are no additional geometric assumptions other than those defining spaces of homogeneous type. To be precise, Auscher and Hytönen proved the following

Theorem A ([AH] Theorem 7.1). *Let (X, d, μ) be a space of homogeneous type in the sense of Coifman and Weiss with quasi-triangle constant A_0 . There exists an orthonormal wavelet basis $\{\psi_\alpha^k\}$, $k \in \mathbb{Z}$, $x_\alpha^k \in \mathcal{Y}^k$, of $L^2(X)$, having exponential decay*

$$(1.8) \quad |\psi_\alpha^k(x)| \leq \frac{C}{\sqrt{\mu(B(x_\alpha^k, \delta^k))}} \exp\left(-\nu\left(\frac{d(x_\alpha^k, x)}{\delta^k}\right)^a\right),$$

Hölder regularity

$$(1.9) \quad |\psi_\alpha^k(x) - \psi_\alpha^k(y)| \leq \frac{C}{\sqrt{\mu(B(x_\alpha^k, \delta^k))}} \left(\frac{d(x, y)}{\delta^k}\right)^\eta \exp\left(-\nu\left(\frac{d(x_\alpha^k, x)}{\delta^k}\right)^a\right)$$

for $d(x, y) \leq \delta^k$, and the cancellation property

$$(1.10) \quad \int_X \psi_\alpha^k(x) d\mu(x) = 0, \quad \text{for } k \in \mathbb{Z}.$$

Moreover, the wavelet expansion is given by

$$(1.11) \quad f(x) = \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{Y}^k} \langle f, \psi_\alpha^k \rangle \psi_\alpha^k(x)$$

in the sense of $L^2(X)$.

Here δ is a fixed small parameter, say $\delta \leq 10^{-3}A_0^{-10}$, $a = (1 + 2\log_2 A_0)^{-1}$, and $C < \infty$, $\nu > 0$ and $\eta \in (0, 1]$ are constants independent of k , α , x and x_α^k . See [AH] for more notations and details of the proof.

Auscher and Hytönen’s orthonormal wavelet bases open the door for developing wavelet analysis on spaces of homogeneous type. For example, applying orthonormal wavelet bases, Auscher and Hytönen [AH] proved the $T(1)$ theorem on spaces of homogeneous type in the sense of Coifman and Weiss. Motivated by Auscher and Hytönen’s work, in [HLW] the Hardy

space theory was developed on space of homogeneous type in the sense of Coifman and Weiss. More precisely, let (X, d, μ) be space of homogeneous type in the sense of Coifman and Weiss with $\mu(X) = \infty$. They first introduce the test function and distribution space as follows.

Definition A ([HLW]). (Test functions) *Fix $x_0 \in X$, $r > 0$, $\beta \in (0, \eta]$ where η is the regularity exponent from Theorem A and $\gamma > 0$. A function f defined on X is said to be a test function of type (x_0, r, β, γ) centered at $x_0 \in X$ if f satisfies the following three conditions.*

(i) (Size condition) *For all $x \in X$,*

$$|f(x)| \leq C \frac{1}{V_r(x_0) + V(x, x_0)} \left(\frac{r}{r + d(x, x_0)} \right)^\gamma.$$

(ii) (Hölder regularity condition) *For all $x, y \in X$ with $d(x, y) < (2A_0)^{-1}(r + d(x, x_0))$,*

$$|f(x) - f(y)| \leq C \left(\frac{d(x, y)}{r + d(x, x_0)} \right)^\beta \frac{1}{V_r(x_0) + V(x, x_0)} \left(\frac{r}{r + d(x, x_0)} \right)^\gamma.$$

(iii) (Cancellation condition)

$$\int_X f(x) d\mu(x) = 0,$$

where $V_r(x_0) = \mu(B(x_0, r))$ and $V(x_0, x) = \mu(B(x_0, d(x, x_0)))$.

Note that, as proved in [HLW], $\frac{\psi_\alpha^k(x)}{\sqrt{\mu(B(x_\alpha^k, \delta^k))}}$ is a test function with $x_0 = x_\alpha^k, r = \delta^k, \beta = \eta$ and any $\gamma > 0$. The test function space is denoted by $G(x_0, r, \beta, \gamma)$, which consists of all test functions of type (x_0, r, β, γ) . The norm of f in $G(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{G(x_0, r, \beta, \gamma)} := \inf\{C > 0 : \text{(i) and (ii) hold}\}.$$

For each fixed x_0 , let $G(\beta, \gamma) := G(x_0, 1, \beta, \gamma)$. It is easy to check that for each fixed $x'_0 \in X$ and $r > 0$, we have $G(x'_0, r, \beta, \gamma) = G(\beta, \gamma)$ with equivalent norms. Furthermore, it is also easy to see that $G(\beta, \gamma)$ is a Banach space with respect to the norm on $G(\beta, \gamma)$.

For $\beta \in (0, \eta]$ and $\gamma > 0$, let $\mathring{G}(\beta, \gamma)$ be the completion of the space $G(\eta, \gamma)$ in the norm of $G(\beta, \gamma)$; of course when $\beta = \eta$ we simply have $\mathring{G}(\beta, \gamma) = \mathring{G}(\eta, \gamma) = G(\eta, \gamma)$. We define the norm on $\mathring{G}(\beta, \gamma)$ by $\|f\|_{\mathring{G}(\beta, \gamma)} := \|f\|_{G(\beta, \gamma)}$.

Definition B ([HLW]). (Distributions) *Fix $x_0 \in X$, $r > 0$, $\beta \in (0, \eta]$ where η is the regularity exponent from Theorem A and $\gamma > 0$. The distribution space $(\mathring{G}(\beta, \gamma))'$ is defined to be the set of all linear functionals \mathcal{L} from $\mathring{G}(\beta, \gamma)$ to \mathbb{C} with the property that there exists $C > 0$ such that for all $f \in \mathring{G}(\beta, \gamma)$,*

$$|\mathcal{L}(f)| \leq C \|f\|_{\mathring{G}(\beta, \gamma)}.$$

A fundamental result proved in [HLW] is the following wavelet representation for test functions and distributions.

Theorem B ([HLW]). (Wavelet reproducing formula for test functions and distributions) Suppose that $f \in \mathring{G}(\beta, \gamma)$ with $\beta, \gamma \in (0, \eta)$. Then the wavelet reproducing formula

$$(1.12) \quad f(x) = \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{Y}^k} \langle f, \psi_\alpha^k \rangle \psi_\alpha^k(x)$$

holds in $\mathring{G}(\beta', \gamma')$ for all $\beta' \in (0, \beta)$ and $\gamma' \in (0, \gamma)$. Moreover, the wavelet reproducing formula (1.12) also holds in the space $(\mathring{G}(\beta, \gamma))'$ of distributions.

Based on the above wavelet reproducing formula, the Littlewood–Paley square function in terms of wavelet coefficients is defined by the following

Definition C ([HLW]). For f in $(\mathring{G}(\beta, \gamma))'$ with $\beta, \gamma \in (0, \eta)$, the discrete Littlewood–Paley square function $S(f)$ of f is defined by

$$(1.13) \quad S(f)(x) := \left\{ \sum_k \sum_{\alpha \in \mathcal{Y}^k} |\langle \psi_\alpha^k, f \rangle \tilde{\chi}_{Q_\alpha^k}(x)|^2 \right\}^{1/2},$$

where $\tilde{\chi}_{Q_\alpha^k}(x) := \chi_{Q_\alpha^k}(x) \mu(Q_\alpha^k)^{-1/2}$ and $\chi_{Q_\alpha^k}(x)$ is the indicator function of the dyadic cube Q_α^k .

The Hardy space on space of homogeneous type in the sense of Coifman and Weiss then is introduced as follows.

Definition D ([HLW]). Suppose that $0 < \beta, \gamma < \eta$ and $\frac{\omega}{\omega + \eta} < p \leq 1$, where η is the regularity given in Theorem A and ω is the upper dimension of X . The Hardy space $H^p(X)$ is defined by

$$H^p(X) := \{f \in (\mathring{G}(\beta, \gamma))' : S(f) \in L^p(X)\}.$$

The norm of $f \in H^p(X)$ is defined by $\|f\|_{H^p(X)} := \|S(f)\|_{L^p(X)}$.

Natural questions arise:

(1) Is the atomic Hardy space $H_{cw}^p(X)$ same as the Hardy space $H^p(X)$ with equivalent norms?

(2) Can one provide a criterion of the boundedness for singular integral operators on these Hardy spaces?

In this paper, we address the above questions. We will give a positive answer for the first question by the following

Theorem 1.1. Let (X, d, μ) be space of homogeneous type in the sense of Coifman and Weiss and $\frac{\omega}{\omega + \eta} < p \leq 1$. Then $H_{cw}^p(X) = H^p(X)$ with equivalent norms. More precisely, if $f \in H_{cw}^p(X)$ then $f \in H^p(X)$ and there exists a constant C such that $\|f\|_{H^p} \leq C\|f\|_{H_{cw}^p}$. Conversely, if $f \in H^p(X)$ then f has an atomic decomposition: $f = \sum_j \lambda_j a_j$ where all a_j 's are $(p, 2)$ atoms and the series converges in the dual of $\mathcal{C}_\alpha(X)$, $\alpha = \frac{1}{p} - 1$. Moreover, $\sum_j |\lambda_j|^p \leq C\|f\|_{H^p}^p$ where the constant C is independent of f .

We would like to point out that the most significant ingredient of Theorem 1.1 is the method of atomic decomposition for subspace $L^2(X) \cap H^p(X)$. More precisely, if $f \in L^2(X) \cap H^p(X)$ then f has an atomic decomposition which converges in both $L^2(X)$ and $H^p(X)$. These facts play a crucial role in this paper. We also remark that if f is a distribution in $(\dot{G}(\beta, \gamma))'$, in general, f may not be a linear functional on $\mathcal{C}_\alpha(X)$. However, Theorem 1.1 implies that if f is a distribution in $(\dot{G}(\beta, \gamma))'$ and belongs to $H^p(X)$, then f can be defined as a linear functional on $\mathcal{C}_\alpha(X)$. One may also observe that the wavelet reproducing formula is not available for providing an atomic decomposition for $H^p(X)$ since the wavelets $\psi_\alpha^k(x)$ have no compact supports. To overcome this problem, a crucial idea is to establish a new kind of Calderón-type reproducing formula. See Proposition 2.5 below for such a new reproducing formula. As mentioned before, Macías and Segovia [MS2] gave the maximal function characterization of the Hardy space only for the so-called normal spaces (X, d, μ) where d satisfies the regularity condition in (1.6) and μ satisfies the condition in (1.7). They proved the relation between the atomic Hardy space $H_{cw}^p(X)$ and the maximal Hardy space $H_{max}^p(X)$ only in the sense that if f in $H_{cw}^p(X)$, denoting by \tilde{f} the restriction of f to E^α , the test function space on normal space (X, d, μ) , then $\mathcal{F}f = \tilde{f}$ defines an injective linear transformation from $H_{cw}^p(X)$ onto the space of the distribution g on E^α such that $g_\gamma^*(x)$ belongs to $L^p(X, d\mu)$ and there exist two positive and finite constants c_1 and c_2 such that

$$c_1 \|f\|_{H_{cw}^p} \leq \left(\int \tilde{f}_\gamma^*(x)^p d\mu(x) \right)^{\frac{1}{p}} \leq c_2 \|f\|_{H_{cw}^p}.$$

See Theorem 5.9 in [MS2] for more details.

In order to provide the criterion of the boundedness for singular integrals on the Hardy spaces, we now define singular integral operator on spaces of homogeneous type in the sense of Coifman and Weiss.

Definition 1.2. We say that T is a singular integral operator on space of homogeneous type (X, d, μ) if T is of the form $T(f)(x) = \int K(x, y)f(y)d\mu(y)$, where $K(x, y)$, the kernel of T , satisfies the following estimates:

$$(1.14) \quad |K(x, y)| \leq \frac{C}{V(x, y)}$$

for all $x \neq y$;

$$(1.15) \quad |K(x, y) - K(x', y)| \leq \frac{C}{V(x, y)} \left(\frac{d(x, x')}{d(x, y)} \right)^\eta$$

for $d(x, x') \leq (2A_0)^{-1}d(x, y)$;

$$(1.16) \quad |K(x, y) - K(x, y')| \leq \frac{C}{V(x, y)} \left(\frac{d(y, y')}{d(x, y)} \right)^\eta$$

for $d(y, y') \leq (2A_0)^{-1}d(x, y)$.

The criterion of the boundedness for singular integrals on the Hardy space $H^p(X)$, i.e., the answer of the second question, is the following

Theorem 1.3. *Suppose that T is a singular integral operator with the kernel $K(x, y)$ satisfying the estimates (1.14) and (1.16), and T is bounded on $L^2(X)$. Then T extends to be a bounded operator on $H^p(X)$, $\frac{\omega}{\omega+\eta} < p \leq 1$, if and only if $T^*(1) = 0$.*

Here, if T is bounded on $L^2(X)$ and $H^p(X)$, then $T^*(1) = 0$ means that

$$\langle f, T^*(1) \rangle = \langle T(f), 1 \rangle = \int T(f)(x) d\mu(x) = 0$$

for all $f \in L^2(X) \cap H^p(X)$, $\frac{\omega}{\omega+\eta} < p \leq 1$.

We would like to remark that by Theorem 1.1, one could state Theorem 1.3 with $H^p(X)$ replaced by $H_{cw}^p(X)$. The reason for not doing this is that the atomic Hardy space $H_{cw}^p(X)$ is not convenient for proving the boundedness of operators. Indeed, if $f \in H_{cw}^p(X)$ has an atomic decomposition $f(x) = \sum_j \lambda_j a_j(x)$, in general, $T(f)(x)$ can not be written as $\sum_j \lambda_j T(a_j)(x)$ even when T is bounded on $L^2(X)$ and $f \in L^2(X) \cap H_{cw}^p(X)$. However, $H^p(X)$ is more convenient to use for proving the boundedness of operators on the Hardy space. This is because $L^2(X) \cap H^p(X)$ is dense in $H^p(X)$ and, as proved in Theorem 1.1, $f \in L^2(X) \cap H^p(X)$ has a nice atomic decomposition which converges in both $L^2(X)$ and $H^p(X)$. Therefore, if T is bounded on $L^2(X)$ one can get $T(f)(x) = \sum_j \lambda_j T(a_j)(x)$. Then it suffices to verify that $T(a)$ is in $H^p(X)$ with the upper bound uniformly for all $(p, 2)$ atoms $a(x)$. And this can be concluded by applying the molecule theory. Note that the molecule theory was developed by Coifman and Weiss for (X, ρ, μ) where ρ is the measure distance, see page 594 in [CW2]. In this paper, we develop the molecule theory for (X, d, μ) with the original quasi metric d and the doubling measure μ , see Theorem 3.2 below. Moreover, the method of atomic decomposition for subspace $L^2(X) \cap H^p(X)$ will also be applied for the proof of the necessary condition that $T^*(1) = 0$ if T is bounded on $L^2(X)$ and $H^p(X)$. Note that this necessary condition on \mathbb{R}^n was obtained directly from the fact that if $f \in L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$ then $\int_{\mathbb{R}^n} f(x) dx = 0$. This last fact follows from the estimate of the Fourier transform for $f \in L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$. Since the Fourier transform is missing on general spaces of homogeneous type, to show $\int_X f(x) d\mu(x) = 0$ for all $f \in L^2(X) \cap H^p(X)$, $\frac{\omega}{\omega+\eta} < p \leq 1$, a new approach used in this paper is to prove the estimate

$$\|f\|_{L^p} \leq C \|f\|_{H^p}$$

for $f \in L^2(X) \cap H^p(X)$ with the constant C independent of the $L^2(X)$ norm of f . This estimate has their own interest and the method of the atomic decomposition of subspace $L^2(X) \cap H^p(X)$ plays a crucial role in the proof for such an estimate. See Proposition 3.3 below for details of the proof.

The last main result in this paper is the boundedness of singular integrals on the dual of the Hardy space. It was well-known that the Campanato space $\mathcal{C}_{\frac{1}{p}-1}^p(X)$, $0 < p \leq 1$, is the dual of the atomic Hardy space $H_{cw}^p(X)$ as well as that $\text{BMO}(X)$ is the dual of $H_{cw}^1(X)$. In

[HLW] the Carleson measure spaces $\text{CMO}^p(X)$ were introduced and it was proved that space $\text{CMO}^p(X)$ is the dual of $H^p(X)$ as well as $\text{CMO}^1(X) = \text{BMO}(X)$ is the dual of $H^1(X)$. We will prove the boundedness of singular integrals on $\text{CMO}^p(X)$. The reason for doing this is that we will show that $L^2(X) \cap \text{CMO}^p(X)$ is a dense subspace of $\text{CMO}^p(X)$ in the weak topology sense. See Lemma 4.1 below. This weak density argument plays a similar role as the subspace $L^2(X) \cap H^p(X)$ does in the proofs of Theorem 1.1 and Theorem 1.3. To be precise, the space $\text{CMO}^p(X)$ is defined by the following

Definition E ([HLW]). *Let (X, d, μ) be space of homogeneous type in the sense of Coifman and Weiss. Suppose that $\frac{\omega}{\omega+\eta} < p \leq 1$, where ω is the upper dimension of X . The Carleson measure space $\text{CMO}^p(X)$ is defined by*

$$\text{CMO}^p(X) := \{f \in (\mathring{G}(\beta, \gamma))' : \mathcal{C}_p(f) < \infty\},$$

where

$$\mathcal{C}_p(f) := \sup_Q \left\{ \frac{1}{\mu(Q)^{\frac{2}{p}-1}} \sum_{k \in \mathbb{Z}, \alpha \in \mathcal{D}^k, Q_\alpha^k \subset Q} |\langle \psi_\alpha^k, f \rangle|^2 \right\}^{1/2},$$

where Q runs over all quasi-metric dyadic balls in the sense of Auscher and Hytönen.

In [HLW], the following duality between $H^p(X)$ and $\text{CMO}^p(X)$ was proved.

Theorem C ([HLW]). *Suppose $\frac{\omega}{\omega+\eta} < p \leq 1$, where ω is the upper dimension of X . Then the Carleson measure space $\text{CMO}^p(X)$ is the dual of the Hardy space $H^p(X)$:*

$$(H^p(X))' = \text{CMO}^p(X).$$

More precisely, if $g \in \text{CMO}^p(X)$ the map ℓ_g given by $\ell_g(f) = \langle f, g \rangle$, defined initially for $f \in \mathring{G}(\beta, \gamma)$, extends to a continuous linear functional on $H^p(X)$ with $\|\ell_g\| \approx \|g\|_{\text{CMO}^p(X)}$. Conversely, for every $\ell \in (H^p(X))^*$ there exists some $g \in \text{CMO}^p(X)$ so that $\ell = \ell_g$.

In particular,

$$(H^1(X))' = \text{BMO}(X) = \text{CMO}^1(X).$$

The last result in this paper is the following

Theorem 1.4. *If T is a singular integral with the kernel $K(x, y)$ satisfying the estimates (1.14) and (1.15), and T is bounded on $L^2(X)$ then T extends to be a bounded operaor on $\text{CMO}^p(X)$ if and only if $T(1) = 0$.*

Here, again, if T is bounded on $L^2(X)$ and $\text{CMO}^p(X)$, $T(1) = 0$ means that $\langle T1, f \rangle = \langle 1, T^*(f) \rangle = \int_X T^*(f)(x) d\mu(x) = 0$ for all $f \in L^2(X) \cap H^p(X)$, $\frac{\omega}{\omega+\eta} < p \leq 1$.

Finally, we will show that $\text{CMO}^p(X) = \mathcal{C}_{\frac{1}{p}-1}(X)$, $\frac{\omega}{\omega+\eta} < p \leq 1$, with the equivalent norms. See Proposition 4.3 below. Hence, by the above theorem, we also provide the boundedness of singular integrals on the Campanato space.

The paper is organized as follows. In Section 2, we establish a new reproducing formula (Proposition 2.5) and then prove Theorem 1.1. In Section 3, we develop the molecule theory

for Hardy spaces $H^p(X)$ (Theorem 3.2) and prove Theorem 1.3. In the last section we show the weak density argument (Lemma 4.1) and give the proof of Theorem 1.4.

2. EQUIVALENCE OF $H_{cw}^p(X)$ AND $H^p(X)$

2.1. The proof for $\|f\|_{H^p} \lesssim \|f\|_{H_{cw}^p}$. To show $\|f\|_{H^p} \lesssim \|f\|_{H_{cw}^p}$, we first need the following

Lemma 2.1. *Suppose that $f \in \mathring{G}(\beta, \gamma)$ with $0 < \beta \leq \eta$, $\frac{\omega}{\omega+\eta} < p \leq 1$, $\gamma > \omega(1/p - 1)$. Then $f \in \mathcal{C}_{\frac{1}{p}-1}^1(X)$. Particularly, the wavelet basis $\psi_\alpha^k(x)$ belongs to $\mathcal{C}_{\frac{1}{p}-1}^1(X)$.*

Proof. Suppose that $f \in \mathring{G}(\beta, \gamma)$ with $\|f\|_{\mathring{G}(\beta, \gamma)} = 1$ and $0 < \beta \leq \eta$, $\frac{\omega}{\omega+\eta} < p \leq 1$, $\gamma > \omega(1/p - 1)$. Let $B = B(x_B, r)$ with $x_B \in X$, $r > 0$ be any fixed quasi-ball. To show that f belongs to $\mathcal{C}_{\frac{1}{p}-1}^1(X)$, we consider that r is large and it is small, where the size and the smoothness conditions on f will be applied, respectively. To be more precise, for large r , this means that $r \geq \frac{1}{4A_0^2}$, we consider two cases: $d(x_B, x_0) \leq 2A_0r$ and $d(x_B, x_0) > 2A_0r$. For the first case, if $y \in B(x_0, 1)$ then $d(x_B, y) \leq A_0(1 + d(x_B, x_0)) \leq A_0(2A_0r + 1)$, by the doubling property on μ , thus $V(x_0, 1) \leq V(x_B, A_0(2A_0r + 1)) \leq C\left(\frac{A_0(2A_0r+1)}{r}\right)^\omega V(x_B, r) \leq CV(x_B, r) = C\mu(B)$ and by the size condition on f , we have

$$\left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^2 d\mu(x)\right)^{1/2} \leq C \frac{\|f\|_{\mathring{G}(\beta, \gamma)}}{V_1(x_0)} \leq C(\mu(B))^{1/p-1} (V_1(x_0))^{-\frac{1}{p}}.$$

For the second case, that is, $r \geq \frac{1}{4A_0^2}$ and $d(x_B, x_0) > 2A_0r$. If $x \in B(x_B, r)$, by the quasi-inequality, $d(x_B, x_0) \leq A_0[d(x_B, x) + d(x, x_0)] \leq A_0[r + d(x, x_0)]$. This together with the fact that $r < \frac{d(x_B, x_0)}{2A_0}$ implies that $d(x, x_0) \geq \frac{1}{2A_0}d(x_B, x_0)$. Similarly, if $y \in B(x_B, r)$ then $d(y, x_0) \geq \frac{1}{2A_0}d(x_B, x_0)$. Therefore, for the second case, by the size condition on f and for all $x, y \in B(x_B, r)$,

$$|f(x) - f(y)| \leq C \frac{1}{V_1(x_0)} \left(\frac{1}{1 + d(x_B, x_0)}\right)^\gamma \|f\|_{\mathring{G}(\beta, \gamma)}.$$

Note that $B(x_0, 1) \subset B(x_B, A_0(1 + d(x_B, x_0)))$ and hence, the doubling property on μ implies that $V(x_0, 1) \leq V(x_B, A_0(1 + d(x_B, x_0))) \leq C\left(\frac{A_0(1+d(x_B, x_0))}{r}\right)^\omega V(x_B, r)$. We obtain that

$$\begin{aligned} & \left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^2 d\mu(x)\right)^{1/2} \\ & \leq C \frac{1}{V_1(x_0)} \left(\frac{1}{1 + d(x_B, x_0)}\right)^\gamma \|f\|_{\mathring{G}(\beta, \gamma)} \\ & \leq C \frac{1}{V_1(x_0)} \left(\frac{1}{1 + d(x_B, x_0)}\right)^\gamma \left(\frac{A_0(1 + d(x_B, x_0))}{r}\right)^{\omega(\frac{1}{p}-1)} \left(\frac{V(x_B, r)}{V(x_0, 1)}\right)^{\frac{1}{p}-1} \\ & \leq C(\mu(B))^{1/p-1} (V_1(x_0))^{-\frac{1}{p}} \end{aligned}$$

since $r > \frac{1}{4A_0^2}$ and $\gamma > \omega(1/p - 1)$.

We now consider the small r , that is, $r < \frac{1}{4A_0^2}$. Note that in this case, if $x, y \in B(x_B, r)$ then $d(x, y) \leq 2A_0r \leq \frac{1}{2A_0}$. Thus, we can apply the smoothness condition on f to get

$$|f(x) - f(y)| \leq C \left(\frac{2A_0r}{1 + d(x, x_0)} \right)^\eta \frac{1}{V_1(x_0)} \left(\frac{1}{1 + d(x, x_0)} \right)^\gamma \|f\|_{\dot{G}(\beta, \gamma)}.$$

Similarly, we consider two cases: $d(x_B, x_0) \leq 2A_0r$ and $d(x_B, x_0) > 2A_0r$. The same conclusions hold, that is, $V(x_0, 1) \leq C \left(\frac{A_0(2A_0r+1)}{r} \right)^\omega V(x_B, r)$ and $d(x, x_0) \geq \frac{1}{2A_0}d(x_B, x_0)$, $V(x_0, 1) \leq C \left(\frac{A_0(1+d(x_B, x_0))}{r} \right)^\omega V(x_B, r)$, respectively for these two cases. Therefore, for the first case, we obtain that

$$\begin{aligned} \left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^2 d\mu(x) \right)^{1/2} &\leq C \frac{\|f\|_{\dot{G}(\beta, \gamma)}}{V_1(x_0)} r^\eta \\ &\leq Cr^\eta \left(\frac{A_0(2A_0r+1)}{r} \right)^{\omega(\frac{1}{p}-1)} (\mu(B))^{1/p-1} (V_1(x_0))^{-\frac{1}{p}} \\ &\leq C(\mu(B))^{1/p-1} (V_1(x_0))^{-\frac{1}{p}} \end{aligned}$$

since $r < \frac{1}{4A_0^2}$ and the condition $\frac{\omega}{\omega+\eta} < p \leq 1$ implies $\eta > \omega(\frac{1}{p} - 1)$.

While for the second case, we have

$$\begin{aligned} \left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^2 d\mu(x) \right)^{1/2} &\leq C \frac{\|f\|_{\dot{G}(\beta, \gamma)}}{V_1(x_0)} \left(\frac{r}{1 + d(x_B, x_0)} \right)^\eta \left(\frac{1}{1 + d(x_B, x_0)} \right)^\gamma \\ &\leq C \left(\frac{r}{1 + d(x_B, x_0)} \right)^\eta \left(\frac{1}{1 + d(x_B, x_0)} \right)^\gamma \left(\frac{A_0(1 + d(x_B, x_0))}{r} \right)^{\omega(\frac{1}{p}-1)} (\mu(B))^{1/p-1} (V_1(x_0))^{-\frac{1}{p}} \\ &\leq C(\mu(B))^{1/p-1} (V_1(x_0))^{-\frac{1}{p}} \end{aligned}$$

since $\eta > \omega(\frac{1}{p} - 1)$ and $\gamma > \omega(\frac{1}{p} - 1)$.

By a result in [HLW], $\psi_\alpha^k(x)/\sqrt{\mu(B(y_\alpha^k, \delta^k))}$ belongs to $\dot{G}(\eta, \gamma)$ with any $\gamma > 0$ and hence $\psi_\alpha^k(x)$ belongs to $\mathcal{C}_{\frac{1}{p}-1}^\gamma(X)$. The proof of Lemma 2.1 is complete. \square

We now prove $\|f\|_{H^p} \lesssim \|f\|_{H_{cw}^p}$. Suppose that $f \in H_{cw}^p(X)$ and $f = \sum_j \lambda_j a_j$ where a_j are $(p, 2)$ atoms, $\sum_j |\lambda_j|^p < \infty$, and the series converges in $(\mathcal{C}_{\frac{1}{p}-1}^\gamma(X))'$. Thus, by the above lemma, $S(f) \leq \sum_j |\lambda_j| |S(a_j)|$ and hence, $\|f\|_{H^p}^p = \|S(f)\|_{L^p}^p \leq \sum_j |\lambda_j|^p \|S(a_j)\|_{L^p}^p$. We claim that for each $(p, 2)$ atom $a(x)$,

$$(2.1) \quad \|S(a)\|_{L^p} \leq C,$$

where the constant C is independent of $a(x)$. The claim then implies that $\|f\|_{H^p} \leq C(\sum_j |\lambda_j|^p)^{\frac{1}{p}}$. Taking the infimum for all representations of f gives the desired result. To verify the claim (2.1) and simplify the calculation, we will apply a result proved in [HLW]. More precisely, we need the following

Definition 2.2. (Continuous square function) Let $D_k(x, y) = \sum_{\alpha \in \mathcal{A}^k} \psi_\alpha^k(x) \psi_\alpha^k(y)$. For $f \in (\dot{G}(\beta, \gamma))'$ with $\beta, \gamma \in (0, \eta)$, the *continuous Littlewood–Paley square function* $S_c(f)$ of f is

defined by

$$S_c(f)(x) := \left\{ \sum_k |D_k(f)(x)|^2 \right\}^{1/2}.$$

The following result was proved in [HLW].

Theorem 2.3. (Littlewood–Paley theory) *Fix $\beta, \gamma \in (0, \eta)$ and $p \in (\frac{\omega}{\omega+\eta}, \infty)$, where ω is the upper dimension of (X, d, μ) . For f in $L^2(X)$, we have*

$$\|S(f)\|_{L^p} \sim \|S_c(f)\|_{L^p}.$$

By Theorem 2.3, to verify the claim in (2.1), it suffices to show

$$(2.2) \quad \|S_c(a)\|_{L^p} \leq C$$

for any $(p, 2)$ atom a and the constant C independent of atoms a . To do this, suppose that a is an $(p, 2)$ atom supported in the ball $B(x_0, r)$. Thus,

$$\begin{aligned} \int_X S_c(a)^p(x) d\mu(x) &= \int_{B(x_0, 2A_0r)} S_c(a)^p(x) d\mu(x) + \int_{B(x_0, 2A_0r)^c} S_c(a)^p(x) d\mu(x) \\ &= I + II. \end{aligned}$$

Applying Hölder inequality, the L^2 boundedness of $S_c(f)$, the size condition on a and the doubling property on the measure μ imply that

$$\begin{aligned} I &\leq C\mu(B(x_0, 2A_0r))^{1-\frac{p}{2}} (\|S_c(a)\|_{L^2}^2)^{\frac{p}{2}} \\ &\leq CA_0^{w(1-\frac{p}{2})} (\mu(B(x_0, r)))^{1-\frac{p}{2}} (\mu(B(x_0, r))^{\frac{1}{2}-\frac{1}{p}})^p \\ &\leq C. \end{aligned}$$

To estimate II , we show that there a constant C such that for $x \in (B(x_0, 2A_0r))^c$,

$$(2.3) \quad S(a)(x) \leq C\mu(B(x_0, r))^{1-\frac{1}{p}} \left(\frac{r}{d(x, x_0)} \right)^\eta \frac{1}{V(x, x_0)}.$$

Assuming the estimate in (2.3) for the moment, then

$$\begin{aligned} II &\leq C\mu(B(x_0, r))^{p-1} r^{p\eta} \int_{B(x_0, 2A_0r)^c} d(x, x_0)^{-p\eta} \left(\frac{1}{V(x, x_0)} \right)^p d\mu(x) \\ &\leq C\mu(B(x_0, r))^{p-1} r^{p\eta} \sum_{k=1}^{\infty} \int_{2^k A_0r < d(x, x_0) \leq 2^{k+1} A_0r} 2^{-kp\eta} r^{-p\eta} \left(\frac{1}{\mu(B(x_0, 2^k A_0r))} \right)^p d\mu(x) \\ &\leq C\mu(B(x_0, r))^{p-1} \sum_{k=1}^{\infty} 2^{-kp\eta} (\mu(B(x_0, 2^k A_0r)))^{1-p} \\ &\leq C\mu(B(x_0, r))^{p-1} \sum_{k=1}^{\infty} 2^{-k(p\eta+\omega(p-1))} (\mu(B(x_0, r)))^{1-p} \\ &\leq C, \end{aligned}$$

where the doubling property on the measure μ and the fact that $\frac{\omega}{\omega+\eta} < p \leq 1$ are used for the last two inequalities, respectively. The proof of claim (2.1) is concluded. Therefore, we

only need to show the estimate in (2.3). Note that, by Lemma 3.6 in [HLW], for fixed x and k , $D_k(x, y)$, as the function of the variable of y , belongs to $\mathring{G}(\eta, \gamma)$ for any $\gamma > 0$. If $x \in (B(x_0, 2A_0r))^c$, using the cancellation condition on a and smoothness condition on the kernel of D_k with the second variable,

$$\begin{aligned} |D_k a(x)| &= \left| \int (D_k(x, y) - D_k(x, x_0)) a(y) d\mu(y) \right| \\ &\leq C \int_{B(x_0, r)} \left(\frac{d(y, x_0)}{\delta^k + d(x, x_0)} \right)^\eta \frac{1}{V_{\delta^k}(x) + V(x, x_0)} \left(\frac{\delta^k}{\delta^k + d(x, x_0)} \right)^\gamma |a(y)| d\mu(y) \\ &\leq C \left(\frac{r}{\delta^k + d(x, x_0)} \right)^\eta \frac{1}{V_{\delta^k}(x) + V(x, x_0)} \left(\frac{\delta^k}{\delta^k + d(x, x_0)} \right)^\gamma \mu(B(x_0, r))^{1-\frac{1}{p}}, \end{aligned}$$

where the facts that $d(y, x_0) \leq r \leq \frac{1}{2A_0}d(x, x_0)$ and the size condition on a are used in the first and the last inequalities, respectively. This implies that if $x \in (B(x_0, 2A_0r))^c$,

$$\begin{aligned} S_c(a)(x) &= \left\{ \sum_k |D_k(a)(x)|^2 \right\}^{1/2} \\ &\leq C \mu(B(x_0, r))^{1-\frac{1}{p}} \left\{ \sum_k \left| \left(\frac{r}{\delta^k + d(x, x_0)} \right)^\eta \frac{1}{V_{\delta^k}(x) + V(x, x_0)} \left(\frac{\delta^k}{\delta^k + d(x, x_0)} \right)^\gamma \right|^2 \right\}^{1/2} \\ &\leq C \mu(B(x_0, r))^{1-\frac{1}{p}} \left(\frac{r}{d(x, x_0)} \right)^\eta \frac{1}{V(x, x_0)} \left\{ \sum_{\delta^k \leq d(x, x_0)} \left(\frac{\delta^k}{d(x, x_0)} \right)^{2\gamma} \right\}^{1/2} \\ &\quad + C \mu(B(x_0, r))^{1-\frac{1}{p}} \frac{r^\eta}{V(x, x_0)} \left\{ \sum_{\delta^k > d(x, x_0)} \delta^{-2\eta k} \right\}^{1/2} \\ &\leq C \mu(B(x_0, r))^{1-\frac{1}{p}} \left(\frac{r}{d(x, x_0)} \right)^\eta \frac{1}{V(x, x_0)}. \end{aligned}$$

The proof of the estimate in (2.3) is concluded and hence, the proof for $\|f\|_{H^p} \lesssim \|f\|_{H_{cw}^p}$ is complete.

2.2. A new Calderón-type reproducing formula. To show $\|f\|_{H_{cw}^p} \lesssim \|f\|_{H^p}$, we observe that the wavelet reproducing formula is not available since the wavelets $\psi_\alpha^k(x)$ have no compact supports. To overcome this problem, a crucial idea is to establish a new kind of Calderón-type reproducing formula. For this purpose, we need a result in [MS1]. To be precise, in [MS1], Macías and Segovia proved that for any space of homogeneous type (X, d, μ) in the sense of Coifman and Weiss, there exists a quasi-metric d' with the Hölder regularity, which is geometrically equivalent to the original quasi-metric d . To be more precise, we state their result as follows.

Theorem 2.4 ([MS1]). *Let $d(x, y)$ be a quasi-metric on a set X . Then there exists a quasi-metric $d'(x, y)$ on X , a finite constant C and a number $\theta \in (0, 1)$ such that*

(i) $d'(x, y)$ is geometrically equivalent to $d(x, y)$, that is, $d'(x, y) \approx d(x, y)$ for all $x, y \in X$, and

(ii) for every x, y in X and $r > 0$,

$$|d'(x, y) - d'(x, z)| \leq Cr^{1-\theta} d'(x, y)^\theta$$

holds whenever $d'(x, y)$ and $d'(x, z)$ are both smaller than r .

We now establish a new Calderón-type reproducing formula on space of homogeneous type (X, d', μ) . To do this, we will apply Coifman's construction for an approximation to the identity on (X, d', μ) . More precisely, let $h \in C^1(\mathbb{R})$ be such that $h(t) = 1$ if $|t| \leq 1$, $h(t) = 0$ if $|t| \geq \delta^{-1}$, and $0 \leq h(t) \leq 1$ for all $t \in \mathbb{R}$. For any $k \in \mathbb{Z}$, we define

$$T_k(f)(x) = \int_X h(\delta^{-k} d'(x, y)) f(y) d\mu(y).$$

Obviously, we have $V_{\delta^k}(x) \leq T_k(1)(x) \leq V_{\delta^{-1+k}}(x)$, that is, $T_k(1)(x) \sim V_{\delta^k}(x)$. By the quasi metric d' and the doubling property on μ , it is easy to see that for any fixed constant c and $r > 0$, if $d'(x, y) \leq cr$ then $V_r(x) \sim V_r(y)$. Therefore,

$$T_k\left(\frac{1}{T_k(1)}\right)(x) = \int_X h(\delta^{-k} d'(x, y)) \frac{1}{T_k(1)(y)} d\mu(y) \sim 1.$$

We now define two multiplication operators by $M_k(f)(x) = (T_k(1)(x))^{-1} f(x)$ and $W_k(f)(x) = (T_k(\frac{1}{T_k(1)})(x))^{-1} f(x)$ and operators $S_k(f)(x) = M_k T_k W_k T_k M_k(f)(x)$. We claim that $S_k(x, y)$, the kernel of S_k , satisfies the following conditions:

- (i) $S_k(x, y) = 0$ for $d'(x, y) \geq C\delta^k$, and $\|S_k\|_\infty \leq C \frac{1}{V_{\delta^k}(x) + V_{\delta^k}(y)}$,
- (ii) $|S_k(x, y) - S_k(x', y)| \leq C \left(\frac{d'(x, x')}{\delta^k}\right)^\theta \frac{1}{V_{\delta^k}(x) + V_{\delta^k}(y)}$,
- (iii) $|S_k(x, y) - S_k(x, y')| \leq C \left(\frac{d'(y, y')}{\delta^k}\right)^\theta \frac{1}{V_{\delta^k}(x) + V_{\delta^k}(y)}$,
- (iv) $|[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]|$
 $\leq C \left(\frac{d'(x, x')}{\delta^k}\right)^\theta \left(\frac{d'(y, y')}{\delta^k}\right)^\theta \frac{1}{V_{\delta^k}(x) + V_{\delta^k}(y)}$,
- (v) $\int_X S_k(x, y) d\mu(y) = \int_X S_k(x, y) d\mu(x) = 1.$

To verify the above claim, note that $S_k(x, y) = S_k(y, x)$ and $S_k(1) = 1$. Thus, (v) holds and we only need to check (i), (ii) and (iv). We first check (i) and write

$$S_k(x, y) = \frac{1}{T_k(1)(x)} \left\{ \int_X h(\delta^{-k} d'(x, z)) \frac{1}{T_k\left(\frac{1}{T_k(1)}\right)(z)} h(\delta^{-k} d'(z, y)) d\mu(z) \right\} \frac{1}{T_k(1)(y)}.$$

By the condition on the support for the function h , $S_k(x, y) \neq 0$, then $d'(x, y) \leq 2A_0\delta^{-1+k}$. This implies (i) with the constant $C = 2A_0(\delta^{-1} + 1)$. To see (ii), by (i) we only need to consider $d'(x, x') \leq \delta^k$. Indeed, if $d'(x, x') > \delta^k$ then (ii) follows directly from (i). To show (ii) for the case that $d'(x, x') \leq \delta^k$, we write

$$S_k(x, y) - S_k(x', y)$$

$$\begin{aligned}
&= \left[\frac{1}{T_k(1)(x)} - \frac{1}{T_k(1)(x')} \right] \left\{ \int_X h(\delta^{-k}d'(x, z)) \frac{1}{T_k\left(\frac{1}{T_k(1)}\right)(z)} h(\delta^{-k}d'(z, y)) d\mu(z) \right\} \frac{1}{T_k(1)(y)} \\
&\quad + \frac{1}{T_k(1)(x')} \\
&\quad \times \left\{ \int_X [h(\delta^{-k}d'(x, z)) - h(\delta^{-k}d'(x', z))] \frac{1}{T_k\left(\frac{1}{T_k(1)}\right)(z)} h(\delta^{-k}d'(z, y)) d\mu(z) \right\} \frac{1}{T_k(1)(y)} \\
&=: Z_1 + Z_2.
\end{aligned}$$

For $d'(x, x') \leq \delta^k$, by the regularity on the function h , we have

$$\left| \frac{1}{T_k(1)(x)} - \frac{1}{T_k(1)(x')} \right| \lesssim \frac{1}{V_{\delta^k}(x)^2} (\delta^{-k}d'(x, x'))^\theta \mu(B(x, 2A_0\delta^{-1+k})),$$

which implies that $|Z_1| \lesssim \left(\frac{d(x, x')}{\delta^k}\right)^\theta \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}$.

For Z_2 , observe that if $d'(x', x) \leq \delta^k$ and $h(\delta^{-k}d'(x, z)) - h(\delta^{-k}d'(x', z)) \neq 0$, then $d(x, z) < A_0(\delta^{-1} + 1)\delta^k$. Thus, by the mean value theorem, we obtain

$$\begin{aligned}
&\left| \int_X [h(\delta^{-k}d'(x, z)) - h(\delta^{-k}d'(x', z))] \frac{1}{T_k\left(\frac{1}{T_k(1)}\right)(z)} h(\delta^{-k}d'(z, y)) d\mu(z) \right| \\
&\lesssim (\delta^{-k}d'(x, x'))^\theta \mu(B(x, A_0(\delta^{-1} + 1)\delta^k)),
\end{aligned}$$

which together with the doubling property on μ implies that

$$|Z_2| \lesssim \frac{(\delta^{-k}d(x, x'))^\theta}{V_{\delta^k}(x)} \sim \left(\frac{d'(x, x')}{\delta^k}\right)^\theta \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}.$$

These estimates for Z_1 and Z_2 implies (ii).

Finally, note that

$$\begin{aligned}
&|[S_k(x, y) - S_k(x', y)] - [S_k(x, y') - S_k(x', y')]| \\
&= \left[\frac{1}{T_k(1)(x)} - \frac{1}{T_k(1)(x')} \right] \\
&\quad \times \left\{ \int_X h(2^k d(x, z)) \frac{1}{T_k\left(\frac{1}{T_k(1)}\right)(z)} h(2^k d(z, y)) d\mu(z) \right\} \left[\frac{1}{T_k(1)(y)} - \frac{1}{T_k(1)(y')} \right].
\end{aligned}$$

Repeating the similar proof for (ii) gives (iv).

We observe that $S_k(x, y)$ have compact support with respect to the quasi metric d' . However, the quasi metric d is geometrically equivalent to d' and hence $S_k(x, y)$ have also compact supports with respect to d . This observation will be used for showing the support condition of atoms in the proof for $\|f\|_{H_{cw}^p} \lesssim \|f\|_{H^p}$.

Now we are ready to establish a new Caderón-type reproducing formula.

Proposition 2.5. *Let $D_k := S_{k+1} - S_k$. Then for each given $f \in L^2(X) \cap H^p(X)$, $\frac{\omega}{\omega+\eta} < p \leq 1$, there exists a unique function $g \in L^2(X) \cap H^p(X)$ such that $\|f\|_{L^2} \sim \|g\|_{L^2}$, $\|f\|_{H^p} \sim \|g\|_{H^p}$*

and

$$(2.4) \quad f(x) = \sum_k \sum_{\alpha \in \mathcal{X}^{k+N}} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}).$$

where the series converges in $L^2(X) \cap H^p(X)$, N is a large fixed integer and $\tilde{D}_k = \sum_{|j| \leq N} D_{k+j}$.

Note that in the wavelet expression given in Theorem A, for each $k \in \mathbb{Z}$ the sum runs over the set $\alpha \in \mathcal{Y}^k$, while in this new Calderón-type reproducing formula, for each $k \in \mathbb{Z}$ the sum runs over the set $\alpha \in \mathcal{X}^{k+N}$. Besides the distinction between \mathcal{Y} and \mathcal{X} , the main difference here is that in the the wavelet expressions the function f is involved on both sides but rather, in this new Calderón-type reproducing formula, functions f and g are involved on both sides, respectively. However, D_k involved in this new reproducing formula has compact support which will be important and used frequently. Finally, in this new reproducing formula, we must sum over all cubes at the smaller scale $k + N$.

We now show Proposition 2.5. Note that the family of operators S_k constructed above is an approximation to the identity in $L^2(X)$. Thus, applying Coifman's decomposition for a fixed positive integer N and $f \in L^2(X)$,

$$\begin{aligned} f(x) &= \sum_l D_l(f)(x) = \sum_l \sum_k D_l D_k(f)(x) \\ &= \sum_k \sum_{l: |k-l| \leq N} D_l D_k(f)(x) + \sum_k \sum_{l: |k-l| > N} D_l D_k(f)(x) \\ &= \sum_k \sum_{\alpha \in \mathcal{X}^{k+N}} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(f)(x_\alpha^{k+N}) \\ &\quad + \left(\sum_k D_k \tilde{D}_k(f)(x) - \sum_k \sum_{\alpha \in \mathcal{X}^{k+N}} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(f)(x_\alpha^{k+N}) \right) \\ &\quad + \sum_k \sum_{l: |k-l| > N} D_k D_l(f)(x) \\ &=: T_N(f)(x) + R_N^{(1)}(f)(x) + R_N^{(2)}(f)(x) \end{aligned}$$

in the sense of $L^2(X)$, where $\tilde{D}_k = \sum_{|j| \leq N} D_{k+j}$. and particularly, as mentioned, the kernel of D_k has compact support.

Note that $T_N = I - R_N^{(1)} - R_N^{(2)}$ by definition and $D_k(x, y)$, the kernels of D_k , satisfy the decay condition (i), particularly, as mentioned, the kernels of D_k have compact supports, the smoothness conditions (ii)–(iv) and the moment conditions $\int_X D_k(x, y) d\mu(y) = \int_X D_k(x, y) d\mu(x) = 0$. Therefore the Cotlar–Stein lemma can be applied to show that $R_N^{(i)}$ as well as T_N are bounded on $L^2(X)$. Moreover, we will show that for $f \in L^2(X)$, $\frac{\omega}{\omega+\eta} < p < \infty$ and $i = 1, 2$,

$$(2.5) \quad \|S(R_N^{(i)}(f))\|_{L^p} \leq C\delta^{\theta N} \|S(f)\|_{L^p}.$$

This will imply that if N is chosen so that $2C\delta^{\theta N} < 1$, then $(T_N)^{-1}$, the inverse of T_N , is bounded on $L^2(X) \cap H^p(X)$. Thus, given $f \in L^2(X) \cap H^p(X)$, set $g = (T_N)^{-1}f$. Then g

satisfies all conditions in Proposition 2.5 and moreover, the representation of f in (2.4) holds. We only prove the estimate in (2.5) for $R_N^{(2)}$ since the proof for $R_N^{(1)}$ is similar to one given in [HLW]. See pages 34–39 in [HLW] for the details. To estimate $\|S(R_N^{(2)}(f))\|_{L^p(X)}$, we write

$$\|S(R_N^{(2)}(f))\|_{L^p} = \left\| \left\{ \sum_k \sum_{\alpha \in \mathcal{Y}^k} |\langle \psi_\alpha^k, R_N^{(2)}(f) \rangle \tilde{\chi}_{Q_\alpha^k}(\cdot)|^2 \right\}^{1/2} \right\|_{L^p}.$$

By the $L^2(X)$ -boundedness of $R_N^{(2)}$ and the wavelet reproducing formula in Theorem A for $f \in L^2(X)$, we have

$$\begin{aligned} \langle \psi_\alpha^k, R_N^{(2)}(f) \rangle &= \sqrt{\mu(Q_\alpha^k)} \left\langle \frac{\psi_\alpha^k}{\sqrt{\mu(Q_\alpha^k)}}, R_N^{(2)}(f) \right\rangle \\ &= \sum_{k'} \sum_{\alpha' \in \mathcal{Y}^{k'}} \sqrt{\mu(Q_\alpha^k)} \sqrt{\mu(Q_{\alpha'}^{k'})} \left\langle \frac{\psi_\alpha^k(\cdot)}{\sqrt{\mu(Q_\alpha^k)}}, \left\langle \sum_{|j-l|>N} D_j D_l(\cdot, \cdot), \frac{\psi_{\alpha'}^{k'}(\cdot)}{\sqrt{\mu(Q_{\alpha'}^{k'})}} \right\rangle \right\rangle \langle \psi_{\alpha'}^{k'}, f \rangle. \end{aligned}$$

To simplify the notation, set $E_k^\alpha(x_\alpha^k, x) = \frac{\psi_\alpha^k(x)}{\sqrt{\mu(Q_\alpha^k)}}$. We now estimate the term

$$\langle E_k^\alpha(\cdot), \left\langle \sum_{|j-l|>N} D_j D_l(\cdot, \cdot), E_{k'}^{\alpha'}(\cdot) \right\rangle \rangle.$$

Observe first that $E_k^\alpha(x_\alpha^k, x)$ is a test function in $\mathring{G}(x_\alpha^k, \delta^k, \eta, \gamma)$ for any $\gamma > 0$ and $D_j(x, y)$ is a test function in $\mathring{G}(x, \delta^j, \theta, \gamma)$ if x is fixed or in $\mathring{G}(y, \delta^j, \theta, \gamma)$ if y is fixed. Furthermore, by standard almost orthogonal estimate, $D_j D_l(x, y)$ satisfies the estimates (i)–(iii) as $D_{j \wedge l}(x, y)$ does with the bounds $C\delta^{|j-l|\theta}$, where, as usual, $j \wedge l = \min\{j, l\}$ denotes the minimum of j and l . Indeed, if $j \leq k$, observing that $D_j D_l(x, y) = \int [D_j(x, z) - D_j(x, y)] D_l(z, y) d\mu(z)$ and applying the smoothness condition on D_j and the size condition on D_l yields that $D_j D_l(x, y)$ satisfies the condition (i) with the constant replaced by $C\delta^{|j-l|\theta}$. Writing $D_j D_l(x, y) - D_j D_l(x', y) = \int \{[D_j(x, z) - D_j(x, y)] - [D_j(x', z) - D_j(x', y)]\} D_l(z, y) d\mu(z) = \int \{[D_j(x, z) - D_j(x', z)] - [D_j(x, y) - D_j(x', y)]\} D_l(z, y) d\mu(z)$ and applying the smoothness condition (iv) on D_j and the size condition on D_l implies that $D_j D_l(x, y)$ satisfies the condition (ii) with the bound replaced by $C\delta^{|j-l|\theta}$. We point out that the condition (iv) was not used in [DJS] for the proof of the L^2 boundedness but this is crucial for the boundedness of $H^p(X)$. We leave the details of the proof to the reader. Set

$$F_{k'}(x, x_{\alpha'}^{k'}) = \sum_{(j,l): |j-l|>N} \langle D_j D_l(x, \cdot), E_{k'}^{\alpha'}(\cdot) \rangle = \sum_{(j,l): |j-l|>N} \int D_j D_l(x, y) E_{k'}^{\alpha'}(y, x_{\alpha'}^{k'}) d\mu(y).$$

We claim that

$$(a) \quad |F_{k'}(x, x_{\alpha'}^{k'})| \leq C\delta^{\theta N} \frac{1}{V_{\delta^{k'}}(x) + V(x, x_{\alpha'}^{k'})} \left(\frac{\delta^{k'}}{\delta^{k'} + d(x, x_{\alpha'}^{k'})} \right)^\gamma,$$

for all $\gamma \in (0, \eta)$, and

$$(b) \quad |F_{k'}(x, x_{\alpha'}^{k'}) - F_{k'}(x', x_{\alpha'}^{k'})|$$

$$\leq C\delta^{\theta N} \left(\frac{d(x, x')}{\delta^{k'}} \right)^{\eta'} \times \left[\frac{1}{V_{\delta^{k'}}(x) + V(x, x_{\alpha'}^{k'})} \left(\frac{\delta^{k'}}{\delta^{k'} + d(x, x_{\alpha'}^{k'})} \right)^{\gamma} + \frac{1}{V_{\delta^{k'}}(x') + V(x', x_{\alpha'}^{k'})} \left(\frac{\delta^{k'}}{\delta^{k'} + d(x', x_{\alpha'}^{k'})} \right)^{\gamma} \right],$$

for all $\gamma, \eta' \in (0, \eta)$.

The main tool to show the above claim is the following almost-orthogonality estimate: There exists a constant C such that

$$(2.6) \quad \left| \left\langle \frac{\psi_{\alpha}^k(\cdot)}{\sqrt{\mu(Q_{\alpha}^k)}}, D_j(\cdot, x) \right\rangle \right| \leq C\delta^{|k-j|\eta} \frac{1}{V_{\delta^{(j \wedge k)}}(x_{\alpha}^k) + V_{\delta^{(j \wedge k)}}(x) + V(x_{\alpha}^k, x)} \left(\frac{\delta^{(j \wedge k)}}{\delta^{(j \wedge k)} + d(x_{\alpha}^k, x)} \right)^{\gamma}.$$

See the proof for such an argument in (4.4) on page 31 [HLW]. Applying the almost-orthogonality estimate in (2.6) with D_j replaced by $D_j D_l$ implies that

$$(2.7) \quad \left| \langle D_j D_l(x, \cdot), E_{k'}^{\alpha'}(\cdot) \rangle \right| \leq C\delta^{|j-l|\theta} \delta^{|k'-j \wedge l|\eta} \frac{1}{V_{\delta^{(j \wedge l \wedge k')}}(x_{\alpha'}^{k'}) + V_{\delta^{(j \wedge l \wedge k')}}(x) + V(x_{\alpha'}^{k'}, x)} \left(\frac{\delta^{(j \wedge l \wedge k')}}{\delta^{(j \wedge l \wedge k')} + d(x_{\alpha'}^{k'}, x)} \right)^{\gamma}.$$

Applying the above estimates for $\eta > \gamma$ gives

$$\begin{aligned} |F_{k'}(x, x_{\alpha'}^{k'})| &= \left| \sum_{(j,l): |j-l| > N} \langle D_j D_l(x, \cdot), E_{k'}^{\alpha'}(\cdot) \rangle \right| \\ &\leq C \sum_{(j,l): |j-l| > N} \delta^{|j-l|\theta} \delta^{|k'-j \wedge l|\eta} \frac{1}{V_{\delta^{(j \wedge l \wedge k')}}(x_{\alpha'}^{k'}) + V_{\delta^{(j \wedge l \wedge k')}}(x) + V(x_{\alpha'}^{k'}, x)} \left(\frac{\delta^{(j \wedge l \wedge k')}}{\delta^{(j \wedge l \wedge k')} + d(x_{\alpha'}^{k'}, x)} \right)^{\gamma} \\ &\leq C \sum_{(j,l): |j-l| > N} \delta^{|j-l|\theta} \delta^{|k'-j \wedge l|(\eta-\gamma)} \frac{1}{V_{\delta^{k'}}(x) + V(x_{\alpha'}^{k'}, x)} \left(\frac{\delta^{k'}}{\delta^{k'} + d(x_{\alpha'}^{k'}, x)} \right)^{\gamma} \\ &\leq C\delta^{N\theta} \frac{1}{V_{\delta^{k'}}(x) + V(x_{\alpha'}^{k'}, x)} \left(\frac{\delta^{k'}}{\delta^{k'} + d(x_{\alpha'}^{k'}, x)} \right)^{\gamma}, \end{aligned}$$

which implies (a). Similarly, for $\eta' < \eta$ and $\gamma < \eta$,

$$\begin{aligned} &|F_{k'}(x, x_{\alpha'}^{k'}) - F_{k'}(x', x_{\alpha'}^{k'})| \\ &= \left| \sum_{(j,l): |j-l| > N} \langle [D_j D_l(x, \cdot) - D_j D_l(x', \cdot)], E_{k'}^{\alpha'}(\cdot) \rangle \right| \\ &\leq C \sum_{(j,l): |j-l| > N} \delta^{|j-l|\theta} \left(\frac{d'(x, x')}{\delta^{j \wedge l}} \right)^{\eta} \delta^{|k'-j \wedge l|\eta} \\ &\quad \times \left[\frac{1}{V_{\delta^{(j \wedge l \wedge k')}}(x_{\alpha'}^{k'}) + V_{\delta^{(j \wedge l \wedge k')}}(x) + V(x_{\alpha'}^{k'}, x)} \left(\frac{\delta^{(j \wedge l \wedge k')}}{\delta^{(j \wedge l \wedge k')} + d(x_{\alpha'}^{k'}, x)} \right)^{\gamma} \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{V_{\delta(j \wedge l \wedge k')}(x_{\alpha'}^{k'}) + V_{\delta(j \wedge l \wedge k')}(x') + V(x_{\alpha'}^{k'}, x')} \left(\frac{\delta^{(j \wedge l \wedge k')}}{\delta^{(j \wedge l \wedge k')} + d(x_{\alpha'}^{k'}, x')} \right)^\gamma \Big] \\
& \leq C \sum_{(j,l): |j-l| > N} \delta^{|j-l|\theta} \left(\frac{d(x, x')}{\delta^{k'}} \right)^{\eta'} \delta^{|k'-j \wedge l|(\eta-\eta')} \times \left[\frac{1}{V_{\delta^{k'}}(x) + V(x_{\alpha'}^{k'}, x)} \left(\frac{\delta^{k'}}{\delta^{k'} + d(x_{\alpha'}^{k'}, x)} \right)^\gamma \right. \\
& \quad \left. + \frac{1}{V_{\delta^{k'}}(x') + V(x_{\alpha'}^{k'}, x')} \left(\frac{\delta^{k'}}{\delta^{k'} + d(x_{\alpha'}^{k'}, x')} \right)^\gamma \right] \\
& \leq C \delta^{N\theta} \left(\frac{d(x, x')}{\delta^{k'}} \right)^{\eta'} \\
& \quad \times \left[\frac{1}{V_{\delta^{k'}}(x) + V(x_{\alpha'}^{k'}, x)} \left(\frac{\delta^{k'}}{\delta^{k'} + d(x_{\alpha'}^{k'}, x)} \right)^\gamma + \frac{1}{V_{\delta^{k'}}(x') + V(x_{\alpha'}^{k'}, x')} \left(\frac{\delta^{k'}}{\delta^{k'} + d(x_{\alpha'}^{k'}, x')} \right)^\gamma \right],
\end{aligned}$$

where the equivalence between d' and d is used in the second inequality. This gives the desired estimate for (b).

Since $F_{k'}(\cdot, x_{k'}^{\alpha'})$ satisfies (a) and (b), from Remark 4.5 in [HLW], we obtain that

$$\begin{aligned}
& \left| \left\langle E_k^\alpha(x_\alpha^k, \cdot), F_{k'}(\cdot, x_{k'}^{\alpha'}) \right\rangle \right| \\
& \leq C \delta^{N\theta} \delta^{|k'-k|\eta''} \frac{1}{V_{\delta(k \wedge k')}(x_{\alpha'}^{k'}) + V_{\delta(k \wedge k')}(x_\alpha^k) + V(x_{\alpha'}^{k'}, x_\alpha^k)} \left(\frac{\delta^{(k \wedge k')}}{\delta^{(k \wedge k')} + d(x_{\alpha'}^{k'}, x_\alpha^k)} \right)^\gamma
\end{aligned}$$

for $\eta'' < \eta'$ and $\gamma < \eta'$.

Thus, we have

$$\begin{aligned}
\left\langle \psi_\alpha^k, R_N^{(2)}(f) \right\rangle &= \sum_{k'} \sum_{\alpha' \in \mathcal{Y}^{k'}} \sqrt{\mu(Q_\alpha^k)} \sqrt{\mu(Q_{\alpha'}^{k'})} \left| \left\langle E_k^\alpha(x_\alpha^k, \cdot), F_{k'}(\cdot, x_{k'}^{\alpha'}) \right\rangle \right| \left| \left\langle \psi_{\alpha'}^{k'}, f \right\rangle \right| \\
&\leq C \delta^{N\theta} \sqrt{\mu(Q_\alpha^k)} \sum_{k'} \sum_{\alpha' \in \mathcal{Y}^{k'}} \frac{1}{V_{\delta(k \wedge k')}(x_{\alpha'}^{k'}) + V_{\delta(k \wedge k')}(x_\alpha^k) + V(x_{\alpha'}^{k'}, x_\alpha^k)} \\
&\quad \times \mu(Q_{\alpha'}^{k'}) \delta^{|k'-k|\eta''} \left(\frac{\delta^{(k \wedge k')}}{\delta^{(k \wedge k')} + d(x_{\alpha'}^{k'}, x_\alpha^k)} \right)^\gamma \left| \left\langle \frac{\psi_{\alpha'}^{k'}}{\sqrt{\mu(Q_{\alpha'}^{k'})}}, f \right\rangle \right|
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
& \left\{ \sum_k \sum_{\alpha \in \mathcal{Y}^k} \left| \left\langle \psi_\alpha^k, R_N^{(2)}(f) \right\rangle \widetilde{\chi}_{Q_\alpha^k}(x) \right|^2 \right\}^{1/2} \\
& \leq C \delta^{N\theta} \left\{ \sum_k \sum_{\alpha \in \mathcal{Y}^k} \left| \sum_{k'} \sum_{\alpha' \in \mathcal{Y}^{k'}} \mu(Q_{\alpha'}^{k'}) \delta^{|k'-k|\eta''} \frac{1}{V_{\delta(k \wedge k')}(x_{\alpha'}^{k'}) + V_{\delta(k \wedge k')}(x_\alpha^k) + V(x_{\alpha'}^{k'}, x_\alpha^k)} \right. \right. \\
& \quad \left. \left. \times \left(\frac{\delta^{(k \wedge k')}}{\delta^{(k \wedge k')} + d(x_{\alpha'}^{k'}, x_\alpha^k)} \right)^\gamma \left| \left\langle \frac{\psi_{\alpha'}^{k'}}{\sqrt{\mu(Q_{\alpha'}^{k'})}}, f \right\rangle \right|^2 \chi_{Q_\alpha^k}(x) \right\}^{1/2}.
\end{aligned}$$

Then, following the same argument as in page 33 of [HLW], i.e., the estimate as in [FJ], pp.147–148 and the Fefferman–Stein vector-valued maximal function inequality in [FS], we

obtain that

$$\begin{aligned}
\|S(R_N^{(2)}(f))\|_{L^p} &= \left\| \left\{ \sum_k \sum_{\alpha \in \mathcal{D}^k} |\langle \psi_\alpha^k, R_N^{(2)}(f) \rangle \tilde{\chi}_{Q_\alpha^k}(\cdot)|^2 \right\}^{1/2} \right\|_{L^p} \\
&\leq C\delta^{N\theta} \left\| \left\{ \sum_{k'} \sum_{\alpha' \in \mathcal{D}^{k'}} |\langle \psi_{\alpha'}^{k'}, f \rangle \tilde{\chi}_{Q_{\alpha'}^{k'}}(\cdot)|^2 \right\}^{1/2} \right\|_{L^p} \\
&= C\delta^{N\theta} \|S(f)\|_{L^p}
\end{aligned}$$

The proof of Proposition 2.5 is concluded.

2.3. The proof for $\|f\|_{H_{cw}^p} \lesssim \|f\|_{H^p}$. Now we show that $\|f\|_{H_{cw}^p} \lesssim \|f\|_{H^p}$ for $f \in L^2(X) \cap H^p(X)$ since $L^2(X) \cap H^p(X)$ is dense in $H^p(X)$. To this end, we define a new Littlewood–Paley square function on $L^2(X) \cap H^p(X)$ by

$$\tilde{S}(f)(x) = \left(\sum_k \sum_{\alpha \in \mathcal{D}^{k+N}} |\tilde{D}_k(g)(x_\alpha^{k+N})|^2 \chi_{Q_\alpha^{k+N}}(x) \right)^{1/2},$$

where f, g are related as given in Proposition 2.5.

By the estimate in (4.2) of Theorem 4.4 in [HLW], we have that for $\frac{\omega}{\omega+\eta} < p \leq \infty$, $\|\tilde{S}(f)\|_{L^p} \leq C\|S(g)\|_{L^p}$. Thus, if $f \in L^2(X) \cap H^p(X)$ and g is given by Proposition 2.5, then $\|\tilde{S}(f)\|_{L^2} \leq C\|g\|_{L^2} \leq C\|f\|_{L^2}$ and $\|\tilde{S}(f)\|_{L^p} \leq C\|g\|_{H^p} \leq C\|f\|_{H^p}$, $\frac{\omega}{\omega+\eta} < p \leq 1$. We now set

$$\Omega_l = \left\{ x \in X : \tilde{S}(f)(x) > 2^l \right\},$$

$$B_l = \left\{ Q_\alpha^k : \mu(Q_\alpha^k \cap \Omega_l) > \frac{1}{2}\mu(Q_\alpha^k) \text{ and } \mu(Q_\alpha^k \cap \Omega_{l+1}) \leq \frac{1}{2}\mu(Q_\alpha^k) \right\}$$

and

$$\tilde{\Omega}_l = \{x \in X : M(\chi_{\Omega_l})(x) > 1/2\},$$

where M is the Hardy–Littlewood maximal function on X and hence, $\mu(\tilde{\Omega}_l) \leq C\mu(\Omega_l)$.

Note that each dyadic cube Q_α^k belongs to one and only one B_l . Applying Proposition 2.5, we can write

$$\begin{aligned}
f(x) &= \sum_l \sum_{Q_\alpha^{k+N} \in B_l} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}) \\
&= \sum_l \sum_{j: \tilde{Q}_\alpha^{j,l} \in B_l} \sum_{\substack{Q_\alpha^{k+N} \in B_l \\ Q_\alpha^{k+N} \subseteq \tilde{Q}_\alpha^{j,l}}} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}),
\end{aligned}$$

where the series converge in $L^2(X)$ and in $H^p(X)$, and we denote $\tilde{Q}_\alpha^{j,l} \in B_l$ by the maximal dyadic cubes contained in B_l .

Finally, we write

$$f(x) = \sum_l \sum_{j: \tilde{Q}_\alpha^{j,l} \in B_l} \lambda_l(\tilde{Q}_\alpha^{j,l}) a_{l, \tilde{Q}_\alpha^{j,l}}(x),$$

where

$$a_{l,\tilde{Q}_\alpha^{j,l}}(x) = \frac{1}{\lambda_l(\tilde{Q}_\alpha^{j,l})} \sum_{\substack{Q_\alpha^{k+N} \in B_l \\ Q_\alpha^{k+N} \subseteq \tilde{Q}_\alpha^{j,l}}} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N})$$

and

$$\lambda_l(\tilde{Q}_\alpha^{j,l}) = \tilde{C} \left(\sum_{\substack{Q_\alpha^{k+N} \in B_l \\ Q_\alpha^{k+N} \subseteq \tilde{Q}_\alpha^{j,l}}} \mu(Q_\alpha^{k+N}) |\tilde{D}_k(g)(x_\alpha^{k+N})|^2 \right)^{1/2} \mu(\tilde{Q}_\alpha^{j,l})^{1/p-1/2}$$

with a constant \tilde{C} to be determined later.

Now we prove that the above decomposition of f is a desired atomic decomposition. To see this, we first show that each $a_{l,\tilde{Q}_\alpha^{j,l}}(x)$ is an $(p, 2)$ atom. Note that the quasi metrics d and d' are geometrically equivalent. This together with the fact, as mentioned, that $D_k(x, x_\alpha^k)$ have compact supports implies that $a_{l,\tilde{Q}_\alpha^{j,l}}(x)$ is supported in $C\tilde{Q}_\alpha^{j,l}$ with $C = C_1 A_0 \delta^{-N} (\delta^{-1} + 1)$, where C_1 is the constant appeared in the equivalence between d and d' . By the cancellation conditions on $D_k(x, x_\alpha^k)$, it is easy to see that

$$\int_X a_{l,\tilde{Q}_\alpha^{j,l}}(x) d\mu(x) = 0.$$

To verify the size condition of $a_{l,\tilde{Q}_\alpha^{j,l}}(x)$, applying the duality argument implies that

$$\begin{aligned} & \|a_{l,\tilde{Q}_\alpha^{j,l}}(x)\|_{L^2} \\ &= \sup_{h \in L^2(X), \|h\|_{L^2}=1} \left| \langle a_{l,\tilde{Q}_\alpha^{j,l}}(x), h(x) \rangle \right| \\ &= \sup_{h \in L^2(X), \|h\|_{L^2}=1} \left| \frac{1}{\lambda_l(\tilde{Q}_\alpha^{j,l})} \sum_{\substack{Q_\alpha^{k+N} \in B_l \\ Q_\alpha^{k+N} \subseteq \tilde{Q}_\alpha^{j,l}}} \mu(Q_\alpha^{k+N}) D_k(h)(x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}) \right| \\ &\leq \sup_{h \in L^2(X), \|h\|_{L^2}=1} \frac{1}{\lambda_l(\tilde{Q}_\alpha^{j,l})} \left(\sum_{\substack{Q_\alpha^{k+N} \in B_l \\ Q_\alpha^{k+N} \subseteq \tilde{Q}_\alpha^{j,l}}} \mu(Q_\alpha^{k+N}) |D_k(h)(x_\alpha^{k+N})|^2 \right)^{1/2} \\ &\quad \times \left(\sum_{\substack{Q_\alpha^{k+N} \in B_l \\ Q_\alpha^{k+N} \subseteq \tilde{Q}_\alpha^{j,l}}} \mu(Q_\alpha^{k+N}) |\tilde{D}_k(g)(x_\alpha^{k+N})|^2 \right)^{1/2} \\ &\leq \sup_{h \in L^2(X), \|h\|_{L^2}=1} \tilde{C} \|h\|_{L^2} \frac{1}{\lambda_l(\tilde{Q}_\alpha^{j,l})} \left(\sum_{\substack{Q_\alpha^{k+N} \in B_l \\ Q_\alpha^{k+N} \subseteq \tilde{Q}_\alpha^{j,l}}} \mu(Q_\alpha^{k+N}) |\tilde{D}_k(g)(x_\alpha^{k+N})|^2 \right)^{1/2} \\ &\leq \mu(\tilde{Q}_\alpha^{j,l})^{1/2-1/p}, \end{aligned}$$

where \tilde{C} is chosen to be the constant satisfying the following estimate:

$$\left(\sum_k \sum_{\alpha \in \mathcal{X}^{k+N}} \mu(Q_\alpha^{k+N}) |D_k(h)(x_\alpha^{k+N})|^2 \right)^{1/2} \leq \tilde{C} \|h\|_{L^2(X)}$$

for each $h \in L^2(X)$.

We finally have

$$\begin{aligned}
& \sum_l \sum_{j: \tilde{Q}_\alpha^{j,l}} |\lambda_{l, \tilde{Q}_\alpha^{j,l}}|^p \\
&= \sum_l \sum_{\tilde{Q}_\alpha^{j,l} \in B_l} \tilde{C}^p \left(\sum_{\substack{Q_\alpha^{k+N} \in B_l \\ Q_\alpha^{k+N} \subseteq \tilde{Q}_\alpha^{j,l}}} \mu(Q_\alpha^{k+N}) |\tilde{D}_k(g)(x_\alpha^{k+N})|^2 \right)^{p/2} \mu(\tilde{Q}_\alpha^{j,l})^{1-p/2} \\
&\leq \tilde{C}^p \sum_l \left(\sum_{\tilde{Q}_\alpha^{j,l} \in B_l} \sum_{\substack{Q_\alpha^{k+N} \in B_l \\ Q_\alpha^{k+N} \subseteq \tilde{Q}_\alpha^{j,l}}} \mu(Q_\alpha^{k+N}) |\tilde{D}_k(g)(x_\alpha^{k+N})|^2 \right)^{p/2} \left(\sum_{\tilde{Q}_\alpha^{j,l} \in B_l} \mu(\tilde{Q}_\alpha^{j,l}) \right)^{1-p/2} \\
&\leq \tilde{C}^p \sum_l \left(\sum_{Q_\alpha^{k+N} \in B_l} \mu(Q_\alpha^{k+N}) |\tilde{D}_k(g)(x_\alpha^{k+N})|^2 \right)^{p/2} \mu(\tilde{\Omega}_l)^{1-p/2},
\end{aligned}$$

where the fact that if $\tilde{Q}_\alpha^{j,l} \in B_l$ then $\tilde{Q}_\alpha^{j,l} \subset \tilde{\Omega}_l$ is used in the last inequality. Note that

$$\int_{\tilde{\Omega}_l \setminus \Omega_{l+1}} \tilde{S}(f)(x)^2 d\mu(x) \leq 2^{(l+1)2} \mu(\tilde{\Omega}_l) \leq C 2^{2l} \mu(\Omega_l).$$

While we also have

$$\begin{aligned}
\int_{\tilde{\Omega}_l \setminus \Omega_{l+1}} \tilde{S}(f)(x)^2 d\mu(x) &= \int_{\tilde{\Omega}_l \setminus \Omega_{l+1}} \sum_k \sum_{\alpha \in \mathcal{X}^{k+N}} |\tilde{D}_k(g)(x_\alpha^{k+N})|^2 \chi_{Q_\alpha^{k+N}}(x) d\mu(x) \\
&\geq \sum_{Q_\alpha^{k+N} \in B_l} |\tilde{D}_k(g)(x_\alpha^{k+N})|^2 \mu(Q_\alpha^{k+N} \cap (\tilde{\Omega}_l \setminus \Omega_{l+1})) \\
&\geq \frac{1}{2} \sum_{Q_\alpha^{k+N} \in B_l} \mu(Q_\alpha^{k+N}) |\tilde{D}_k(g)(x_\alpha^{k+N})|^2.
\end{aligned}$$

As a consequence, we obtain

$$\sum_{Q_\alpha^k \in B_l} \mu(Q_\alpha^k) |\tilde{D}_k(g)(x_\alpha^k)|^2 \leq C 2^{2l} \mu(\Omega_l),$$

which implies that

$$\begin{aligned}
\sum_l \sum_{j: \tilde{Q}_\alpha^{j,l}} |\lambda_{l, \tilde{Q}_\alpha^{j,l}}|^p &\leq \tilde{C}^p \sum_l 2^{lp} \mu(\Omega_l)^{p/2} \mu(\Omega_l)^{1-p/2} \\
&\leq C \|\tilde{S}(f)\|_{L^p}^p \leq C \|f\|_{H^p}^p.
\end{aligned}$$

We have proved that if $f \in L^2(X) \cap H^p(X)$, then f has an atomic decomposition. But we still need to show that the atomic decomposition for f obtained above must converge in the dual of $\mathcal{C}_{\frac{1}{p}-1}(X)$. This follows from the following

Lemma 2.6. *Suppose that (X, d, μ) is space of homogeneous type in the sense of Coifman and Weiss. Let ω is the upper dimension of (X, d, μ) and $\frac{\omega}{\omega+\eta} < p \leq 1$, where η is the Hölder*

exponent of wavelets ψ_α^k . Then $\mathcal{C}_{\frac{1}{p}-1}(X) \subset \text{CMO}^p(X)$. Moreover, there exists a constant C such that

$$\|f\|_{\text{CMO}^p} \leq C \|f\|_{\mathcal{C}_{\frac{1}{p}-1}}.$$

Assuming this lemma for the moment, we show a general argument that if f has an atomic decomposition which converges in the norm of $H^p(X)$ then this decomposition also converges in the dual of $\mathcal{C}_{\frac{1}{p}-1}(X)$. Indeed, suppose that f has an atomic decomposition with $f = \sum_{j \geq 1} \lambda_j a_j(x)$ where the series converges in the norm of $H^p(X)$. Let g be in $\mathcal{C}_{\frac{1}{p}-1}(X)$. By Lemma 2.6, $g \in \text{CMO}^p(X)$ and $\|g\|_{\text{CMO}^p} \leq C \|g\|_{\mathcal{C}_{\frac{1}{p}-1}}$. Thus, by the duality in Theorem C,

$$|\langle f - \sum_{j \leq N} \lambda_j a_j, g \rangle| \leq C \|f - \sum_{j \leq N} \lambda_j a_j\|_{H^p} \|g\|_{\text{CMO}^p} \leq C \|f - \sum_{|j| \leq N} \lambda_j a_j\|_{H^p} \|g\|_{\mathcal{C}_{\frac{1}{p}-1}},$$

where the last term tends to zero as N tends to infinity since $\|f - \sum_{j \leq N} \lambda_j a_j\|_{H^p}$ tends to zero as N tends to infinity. This implies that $\sum_{j \geq 1} \lambda_j a_j(x)$ converges to f in the dual of $\mathcal{C}_{\frac{1}{p}-1}(X)$. Note that what we have proved is that if $f \in H^p(X)$ then f has an atomic decomposition and moreover, this decomposition converges in both of the norm of $L^2(X)$ and the norm of $H^p(X)$. Therefore, by the above general argument, the atomic decomposition of f also converges in the dual of $\mathcal{C}_{\frac{1}{p}-1}(X)$ and hence f belongs to the atomic Hardy space $H_{cw}^p(X)$ in the sense of Coifman and Weiss.

We now prove Lemma 2.6. The idea of the proof of Lemma 2.6 is similar to the proof on \mathbb{R}^n given by C. Fefferman [FS].

Proof of Lemma 2.6. For any fixed quasi-dyadic ball $Q = Q_{\alpha_0}^{k_0}$ in the sense of Auscher and Hytönen, let $Q^* = 2A_0 Q$, the ball centered at the same center as Q with the radius as $2A_0$ times of Q . We denote $f_Q = \frac{1}{\mu(Q)} \int_Q f(x) d\mu(x)$ and write $f = f_1 + f_2 + f_3$, where $f_1 = (f - f_{Q^*})\chi_{Q^*}$, $f_2 = (f - f_{Q^*})\chi_{(Q^*)^c}$ and $f_3 = f_{Q^*}$. Then $\langle f, \psi_\alpha^k \rangle = \langle f_1, \psi_\alpha^k \rangle + \langle f_2, \psi_\alpha^k \rangle$ since $\int_X \psi_\alpha^k(x) d\mu(x) = 0$. This implies that $\|f\|_{\text{CMO}^p} \lesssim \|f_1\|_{\text{CMO}^p} + \|f_2\|_{\text{CMO}^p}$.

The estimate for $\|f_1\|_{\text{CMO}^p}$ follows directly from the following

$$\begin{aligned} \left\{ \frac{1}{\mu(Q)^{\frac{2}{p}-1}} \sum_{k \in \mathbb{Z}, \alpha \in \mathcal{U}^k, Q_\alpha^k \subset Q} |\langle \psi_\alpha^k, f_1 \rangle|^2 \right\}^{1/2} &\lesssim \frac{1}{\mu(Q)^{\frac{1}{p}-\frac{1}{2}}} \left\{ \int_X |f_1(x)|^2 d\mu(x) \right\}^{1/2} \\ &\lesssim \frac{1}{\mu(Q)^{\frac{1}{p}-1}} \left\{ \frac{1}{\mu(Q)} \int_{Q^*} |f(x) - f_{Q^*}|^2 d\mu(x) \right\}^{\frac{1}{2}} \\ &\lesssim C \|f\|_{\mathcal{C}_{\frac{1}{p}-1}}, \end{aligned}$$

where the doubling property on μ is used. To estimate $\|f_2\|_{\text{CMO}^p(X)}$, observe that, by a result in [HLW], $\mu(Q_\alpha^k)^{-\frac{1}{2}} \psi_\alpha^k \in \mathring{G}(\eta, \gamma)$ for any $\gamma > 0$, particularly, we will take $\gamma > \omega(\frac{2}{p} - 2)$. Thus,

$$|\langle f_2, \psi_\alpha^k \rangle| \lesssim \mu(Q_\alpha^k)^{\frac{1}{2}} \int_{(Q^*)^c} |f(x) - f_{Q^*}| \frac{1}{V_{\delta^k}(x) + V(x, x_\alpha^k)} \left(\frac{\delta^k}{\delta^k + d(x, x_\alpha^k)} \right)^\gamma d\mu(x).$$

Applying Hölder inequality together with the fact that

$$\int_X \frac{1}{V_{\delta^k}(x) + V(x, x_\alpha^k)} \left(\frac{\delta^k}{\delta^k + d(x, x_\alpha^k)} \right)^\gamma d\mu(x) \leq C$$

implies that

$$|\langle f_2, \psi_\alpha^k \rangle|^2 \lesssim \mu(Q_\alpha^k) \int_{(Q^*)^c} |f(x) - f_{Q^*}|^2 \frac{1}{V_{\delta^k}(x) + V(x, x_\alpha^k)} \left(\frac{\delta^k}{\delta^k + d(x, x_\alpha^k)} \right)^\gamma d\mu(x).$$

Next, as is easily seen, if $x \in (Q^*)^c$ and $Q_\alpha^k \subset Q$

$$\frac{1}{V_{\delta^k}(x) + V(x, x_\alpha^k)} \left(\frac{\delta^k}{\delta^k + d(x, x_\alpha^k)} \right)^\gamma \lesssim \frac{1}{V(x_{\alpha_0}^{k_0}, x)} \left(\frac{\delta^{k_0}}{\delta^{k_0} + d(x_{\alpha_0}^{k_0}, x)} \right)^\gamma.$$

Therefore, we obtain

$$\begin{aligned} & \left\{ \frac{1}{\mu(Q)^{\frac{2}{p}-1}} \sum_{k \in \mathbb{Z}, \alpha \in \mathcal{Y}^k, Q_\alpha^k \subset Q} |\langle \psi_\alpha^k, f_2 \rangle|^2 \right\}^{1/2} \\ & \lesssim \left\{ \frac{1}{\mu(Q)^{\frac{2}{p}-1}} \sum_{k=k_0}^\infty \sum_{\alpha \in \mathcal{Y}^k: Q_\alpha^k \subset Q} \mu(Q_\alpha^k) \int_{(Q^*)^c} |f(x) - f_{Q^*}|^2 \frac{1}{V(x_{\alpha_0}^{k_0}, x)} \left(\frac{\delta^{k_0}}{\delta^{k_0} + d(x_{\alpha_0}^{k_0}, x)} \right)^\gamma d\mu(x) \right\}^{1/2} \\ & \lesssim \left\{ \sum_{k=k_0}^\infty \frac{\delta^{\gamma(k-k_0)}}{\mu(Q)^{\frac{2}{p}-2}} \int_{(Q^*)^c} |f(x) - f_{Q^*}|^2 \frac{1}{V(x_{\alpha_0}^{k_0}, x)} \left(\frac{\delta^{k_0}}{\delta^{k_0} + d(x_{\alpha_0}^{k_0}, x)} \right)^\gamma d\mu(x) \right\}^{1/2}. \end{aligned}$$

We claim that

$$\left\{ \frac{1}{\mu(Q)^{\frac{2}{p}-2}} \int_{(Q^*)^c} |f(x) - f_{Q^*}|^2 \frac{1}{V(x_{\alpha_0}^{k_0}, x)} \left(\frac{\delta^{k_0}}{\delta^{k_0} + d(x_{\alpha_0}^{k_0}, x)} \right)^\gamma d\mu(x) \right\}^{1/2} \lesssim \|f\|_{C_{\frac{1}{p}-1}},$$

which gives the desired estimate for $\|f_2\|_{\text{CMOP}(X)}$ and hence the proof of Lemma 2.6 is concluded. To verify the claim, let $Q_j^{**} = \delta^{-j-1}Q^*$ for $j \in \mathbb{N}$, we have

$$\begin{aligned} & \left\{ \frac{1}{\mu(Q)^{\frac{2}{p}-2}} \int_{(Q^*)^c} |f(x) - f_{Q^*}|^2 \frac{1}{V(x_{\alpha_0}^{k_0}, x)} \left(\frac{\delta^{k_0}}{\delta^{k_0} + d(x_{\alpha_0}^{k_0}, x)} \right)^\gamma d\mu(x) \right\}^{1/2} \\ & \lesssim \left\{ \frac{1}{\mu(Q)^{\frac{2}{p}-2}} \sum_{j=0}^\infty \delta^{j\gamma} \int_{\{x: 2A_0\delta^{k_0-j} \leq d(x, x_{\alpha_0}^{k_0}) < 2A_0\delta^{k_0-j-1}\}} |f(x) - f_{Q_j^{**}}|^2 \frac{1}{V(x_{\alpha_0}^{k_0}, x)} d\mu(x) \right\}^{1/2} \\ & + \left\{ \frac{1}{\mu(Q)^{\frac{2}{p}-2}} \sum_{j=0}^\infty \delta^{j\gamma} \int_{\{x: 2A_0\delta^{k_0-j} \leq d(x, x_{\alpha_0}^{k_0}) < 2A_0\delta^{k_0-j-1}\}} |f_{Q_j^{**}} - f_{Q^*}|^2 \frac{1}{V(x_{\alpha_0}^{k_0}, x)} d\mu(x) \right\}^{1/2} \\ & =: I + II. \end{aligned}$$

For I , by the doubling property on μ , we have

$$\begin{aligned} I & \lesssim \left\{ \frac{1}{\mu(Q)^{\frac{2}{p}-2}} \sum_{j=0}^\infty \delta^{j\gamma} \frac{1}{\mu(Q_j^{**})} \int_{Q_j^{**}} |f(x) - f_{Q_j^{**}}|^2 d\mu(x) \right\}^{1/2} \\ & \lesssim \left\{ \sum_{j=0}^\infty \delta^{j\gamma} \frac{\mu(Q_j^{**})^{\frac{2}{p}-2}}{\mu(Q)^{\frac{2}{p}-2}} \right\}^{1/2} \|f\|_{C_{\frac{1}{p}-1}} \end{aligned}$$

$$\begin{aligned}
&\lesssim \left\{ \sum_{j=0}^{\infty} \delta^{j\gamma} \delta^{-j\omega\left(\frac{2}{p}-2\right)} \right\}^{1/2} \|f\|_{C^{\frac{1}{p}-1}} \\
&\lesssim \|f\|_{C^{\frac{1}{p}-1}}
\end{aligned}$$

since $\gamma > \omega(\frac{2}{p} - 2)$. The estimate for II follows directly from the doubling property on μ . Indeed,

$$\begin{aligned}
II &\lesssim \left\{ \frac{1}{\mu(Q)^{\frac{2}{p}-2}} \sum_{j=0}^{\infty} \delta^{j\gamma} |f_{Q_j^{**}} - f_{Q^*}|^2 \frac{\mu(Q_j^{**})}{\mu(Q_{j-1}^{**})} \right\}^{1/2} \\
&\lesssim \left\{ \frac{1}{\mu(Q)^{\frac{2}{p}-2}} \sum_{j=0}^{\infty} \delta^{j\gamma} \frac{1}{\mu(Q^*)} \int_{Q^*} |f(x) - f_{Q_j^{**}}|^2 d\mu(x) \right\}^{1/2} \\
&\lesssim \left\{ \sum_{j=0}^{\infty} \delta^{j\gamma} \frac{\mu(Q_j^{**})^{\frac{2}{p}-1}}{\mu(Q)^{\frac{2}{p}-2}} \frac{1}{\mu(Q^*)} \frac{1}{\mu(Q_j^{**})^{\frac{2}{p}-1}} \int_{Q_j^{**}} |f(x) - f_{Q_j^{**}}|^2 d\mu(x) \right\}^{1/2} \\
&\lesssim \left\{ \sum_{j=0}^{\infty} \delta^{j\gamma} \delta^{-j\omega\left(\frac{2}{p}-2\right)} \right\}^{1/2} \|f\|_{C^{\frac{1}{p}-1}} \\
&\lesssim \|f\|_{C^{\frac{1}{p}-1}},
\end{aligned}$$

where the fact that $\gamma > \omega(\frac{2}{p} - 2)$ is used. The proof of the claim is complete and hence the proof of Lemma 2.6 is concluded. \square

3. CRITERION OF THE BOUNDEDNESS FOR SINGULAR INTEGRALS ON THE HARDY SPACES

3.1. Molecule theory on spaces of homogeneous type. As mentioned in Section 1, we develop the theory of molecule on spaces of homogeneous type in the sense of Coifman and Weiss. Let (X, d, μ) be a space of homogeneous type. We define the molecule which depends only on the measure μ . Since we do not have any conditions on the measure other than the doubling condition, we applying a stopping time argument in proving the molecule theory, which is new comparing to all the previous related versions of molecule theory.

Definition 3.1. Suppose $\frac{\omega}{\omega+\eta} < p \leq 1$. A function $m(x) \in L^2(X)$ is said to be a $(p, 2, \epsilon)$ molecule if $\epsilon > 0$, $\frac{\omega}{\omega+\eta-\epsilon} < p \leq 1$ and

$$(3.1) \quad \left(\int_X m(x)^2 d\mu(x) \right) \left(\int_X m(x)^2 V(x, x_0)^{1+\frac{2\eta-2\epsilon}{\omega}} d\mu(x) \right)^{\left(\frac{\omega+2\eta-2\epsilon}{\omega} \frac{p}{2-p} - 1\right)^{-1}} \leq 1,$$

where x_0 is a fixed point in X , ω is the upper dimension of μ .

Note that the fact that $\frac{\omega}{\omega+\eta-\epsilon} < p$ implies $\frac{\omega+2\eta-2\epsilon}{\omega} \frac{p}{2-p} - 1 > 0$ and moreover, the quasi metric d is not used in the definition of molecules. Next we show that each $(p, 2, \epsilon)$ molecule $m(x)$ belongs to $H^p(X)$.

Theorem 3.2. Suppose that m is an $(p, 2, \epsilon)$ molecule. Then $m \in H^p(X)$ and moreover

$$\|m\|_{H^p} \leq C,$$

where the constant C is independent of m .

Proof. The basic idea of the proof is to decompose m into a sum of atoms. For this purpose, we first set $\sigma = \|m\|_{L^2}^{-\alpha}$, where $\alpha = \frac{2p}{(2-p)\omega}$.

First, we point out that $B(x_0, 2^{i+1}\sigma) \rightarrow X$ when i tends to $+\infty$, and that $\mu(X) = +\infty$. Thus, there exists an integer i_0 such that

$$(3.2) \quad \mu(B(x_0, 2^{i_0+1}\sigma)) > \sigma^\omega \quad \text{and} \quad \mu(B(x_0, 2^{i_0}\sigma)) \leq \sigma^\omega.$$

Set

$$\chi_0 = B(x_0, 2^{i_0}\sigma) = \{x \in X : d(x, x_0) < 2^{i_0}\sigma\}$$

and for $i \geq 1$,

$$\chi_i = B(x_0, 2^i 2^{i_0}\sigma) \setminus B(x_0, 2^{i-1} 2^{i_0}\sigma) = \{x \in X : 2^{i-1} 2^{i_0}\sigma \leq d(x, x_0) < 2^i 2^{i_0}\sigma\}.$$

Let $\chi_i(x)$ be the characteristic function on χ_i , $i = 0, 1, 2, \dots$

We claim that there exists an integer $j_1 \geq 1$ such that

$$\mu\left(\bigcup_{\ell=1}^{j_1} \chi_\ell\right) > \mu(\chi_0)$$

and

$$\mu\left(\bigcup_{\ell=1}^{j_1-1} \chi_\ell\right) \leq \mu(\chi_0). \quad \text{If } j_1 = 1, \text{ then this does not apply.}$$

To verify the claim, suppose that such j_1 does not exist. Then for every integer $j > 1$, we should have $\mu\left(\bigcup_{\ell=1}^j \chi_\ell\right) \leq \mu(\chi_0)$. This implies that $\mu(X) = \mu\left(\lim_{j \rightarrow \infty} \bigcup_{\ell=1}^j \chi_\ell\right) = \lim_{j \rightarrow \infty} \mu\left(\bigcup_{\ell=1}^j \chi_\ell\right) \leq \mu(\chi_0)$ and it contradicts with the fact that $\mu(X) = \infty$. Applying the same stopping time argument yields that there exists a sequence $\{j_k\}_k$ such that $j_k > j_{k-1}$ and

$$\mu\left(\bigcup_{\ell=j_k+1}^{j_{k+1}} \chi_\ell\right) > \mu(B(x_0, 2^{j_k} 2^{i_0}\sigma))$$

and

$$\mu\left(\bigcup_{\ell=j_k+1}^{j_{k+1}-1} \chi_\ell\right) \leq \mu(B(x_0, 2^{j_k} 2^{i_0}\sigma)). \quad \text{If } j_{k+1} = j_k + 1, \text{ then this does not apply.}$$

Observe that

$$(3.3) \quad \mu(B(x_0, 2^{j_{k+1}} 2^{i_0}\sigma)) = \mu(B(x_0, 2^{j_k} 2^{i_0}\sigma)) + \mu\left(\bigcup_{\ell=j_k+1}^{j_{k+1}} \chi_\ell\right) \geq 2\mu(B(x_0, 2^{j_k} 2^{i_0}\sigma))$$

for each integer $k \geq 0$. Here we set $j_0 = 0$.

Applying (3.3) and induction yields

$$(3.4) \quad \mu(B(x_0, 2^{j_k} 2^{i_0}\sigma)) \geq 2^k \mu(B(x_0, 2^{i_0}\sigma)) \geq 2^k C_\mu^{-1} \mu(B(x_0, 2^{i_0+1}\sigma)) \geq 2^k C_\mu^{-1} \sigma^\omega,$$

where the second inequality follows from the doubling condition of the measure μ and the last inequality follows from the definition of the integer i_0 , see (3.2).

We point out that for each integer $k \geq 1$, if $j_k = j_{k-1} + 1$, then we directly obtain that $\mu(B(x_0, 2^{j_k} 2^{i_0} \sigma)) \leq C_\mu \mu(B(x_0, 2^{j_{k-1}} 2^{i_0} \sigma))$ from the doubling property of the measure μ . While if $j_k > j_{k-1} + 1$, then

$$\begin{aligned} \mu(B(x_0, 2^{j_k} 2^{i_0} \sigma)) &\leq C_\mu \mu(B(x_0, 2^{j_{k-1}} 2^{i_0} \sigma)) \\ &= C_\mu \mu(B(x_0, 2^{j_{k-1}} 2^{i_0} \sigma) \setminus B(x_0, 2^{j_{k-1}-1} 2^{i_0} \sigma)) + C_\mu \mu(B(x_0, 2^{j_{k-1}-1} 2^{i_0} \sigma)). \end{aligned}$$

Note that

$$\mu(B(x_0, 2^{j_k} 2^{i_0} \sigma) \setminus B(x_0, 2^{j_{k-1}} 2^{i_0} \sigma)) = \mu\left(\bigcup_{\ell=j_{k-1}+1}^{j_k-1} \chi_\ell\right) \leq \mu(B(x_0, 2^{j_{k-1}} 2^{i_0} \sigma)),$$

which, together with the above estimate for the case $j_k = j_{k-1} + 1$, yields

$$(3.5) \quad \mu(B(x_0, 2^{j_k} 2^{i_0} \sigma)) \leq 2C_\mu \mu(B(x_0, 2^{j_{k-1}} 2^{i_0} \sigma))$$

for each integer $k \geq 1$.

We also point out that, by (3.5), we obtain

$$\mu(B(x_0, 2^{j_{k+1}} 2^{i_0} \sigma)) \leq 2C_\mu \mu(B(x_0, 2^{j_k} 2^{i_0} \sigma)),$$

which together with the following estimates

$$\begin{aligned} \mu(B(x_0, 2^{j_k} 2^{i_0} \sigma)) &\leq \mu(B(x_0, 2^{j_{k-1}} 2^{i_0} \sigma)) + \mu\left(\bigcup_{\ell=j_{k-1}+1}^{j_k} \chi_\ell\right) \\ &\leq 2\mu\left(\bigcup_{\ell=j_{k-1}+1}^{j_k} \chi_\ell\right) \end{aligned}$$

gives

$$(3.6) \quad \mu(B(x_0, 2^{j_{k+1}} 2^{i_0} \sigma)) \leq 4C_\mu \mu\left(\bigcup_{\ell=j_{k-1}+1}^{j_k} \chi_\ell\right).$$

We now set

$$\tilde{\chi}_0(x) := \chi_0(x), \quad \tilde{\chi}_{j_k}(x) := \sum_{\ell=j_{k-1}+1}^{j_k} \chi_\ell(x).$$

for integer $k \geq 1$, and

$$(3.7) \quad m_k(x) := m(x) \tilde{\chi}_{j_k}(x) - \frac{1}{\int \tilde{\chi}_{j_k}(z) d\mu(z)} \int_{\tilde{\chi}_{j_k}} m(y) d\mu(y) \tilde{\chi}_{j_k}(x)$$

for each integer $k \geq 0$.

Decompose m by

$$m(x) := \sum_{k=0}^{\infty} m_k(x) + \sum_{k=0}^{\infty} \bar{m}_k \tilde{\chi}_{j_k}(x),$$

where $\overline{m}_k = \int m(x) \tilde{\chi}_{j_k}(x) d\mu(x)$ and $\tilde{\chi}_{j_k}(x) = \frac{\tilde{\chi}_{j_k}(x)}{\int \tilde{\chi}_{j_k}(y) d\mu(y)}$.

To see that $\sum_{k=0}^{\infty} m_k(x)$ gives an atomic decomposition, we will show that m_k are $(p, 2)$ atoms due to multiplication of certain constant. Note that m_k is supported in $\tilde{\chi}_{j_k} = \bigcup_{\ell=j_{k-1}+1}^{j_k} \chi_{\ell}$, and that $\int m_k(x) d\mu(x) = 0$. Therefore, we only need to estimate the L^2 norm of m_k . First, we have

$$\begin{aligned} \|m_0\|_{L^2} &\leq \left(\int_{\chi_0} |m(x)|^2 d\mu(x) \right)^{1/2} + \left(\int_{\chi_0} \left| \frac{1}{\int \chi_0(z) d\mu(z)} \int_{\chi_0} m(y) d\mu(y) \tilde{\chi}_0(x) \right|^2 d\mu(x) \right)^{1/2} \\ &\leq 2 \left(\int_{\chi_0} |m(x)|^2 d\mu(x) \right)^{1/2} \\ &\leq 2 \|m\|_{L^2} \\ &= 2\sigma^{-\frac{1}{\alpha}} \\ &\leq 2\mu(B(x_0, 2^{i_0}\sigma))^{-\frac{1}{\alpha\omega}} \\ &= 2\mu(\chi_0)^{\frac{1}{2}-\frac{1}{p}}, \end{aligned}$$

where in the last inequality we use the fact in (3.2), namely that $\mu(B(x_0, 2^{i_0}\sigma)) \leq \sigma^{\omega}$.

Thus, $2^{-1}m_0(x)$ is a $(p, 2)$ atom.

Similarly, for each $k \geq 1$,

$$\begin{aligned} \|m_k\|_{L^2} &\leq 2 \left(\int_{\tilde{\chi}_{j_k}} |m(x)|^2 d\mu(x) \right)^{1/2} \\ &= 2 \left(\int_{\tilde{\chi}_{j_k}} |m(x)|^2 V(x, x_0)^{1+\frac{2\eta-2\epsilon}{\omega}} V(x, x_0)^{-(1+\frac{2\eta-2\epsilon}{\omega})} d\mu(x) \right)^{1/2} \\ &\leq 2V(x_0, 2^{j_{k-1}}2^{i_0}\sigma)^{-\frac{1}{2}-\frac{\eta-\epsilon}{\omega}} \left(\int_{\tilde{\chi}_{j_k}} |m(x)|^2 V(x, x_0)^{1+\frac{2\eta-2\epsilon}{\omega}} d\mu(x) \right)^{1/2} \\ &\leq 2V(x_0, 2^{j_{k-1}}2^{i_0}\sigma)^{-\frac{1}{2}-\frac{\eta-\epsilon}{\omega}} \|m\|_{L^2}^{-\left(\frac{\omega+2\eta-2\epsilon}{\omega} \frac{p}{2-p}-1\right)} \\ &= 2V(x_0, 2^{j_{k-1}}2^{i_0}\sigma)^{-\frac{1}{2}-\frac{\eta-\epsilon}{\omega}} \sigma^{\frac{1}{\alpha}\left(\frac{\omega+2\eta-2\epsilon}{\omega} \frac{p}{2-p}-1\right)}. \end{aligned}$$

Applying the estimates given in (3.4) yields

$$\sigma^{\frac{1}{\alpha}\left(\frac{\omega+2\eta-2\epsilon}{\omega} \frac{p}{2-p}-1\right)} \leq (C_{\mu} 2^{-k} V(x_0, 2^{j_k} 2^{i_0} \sigma))^{\frac{1}{\omega\alpha}\left(\frac{\omega+2\eta-2\epsilon}{\omega} \frac{p}{2-p}-1\right)},$$

which, together with the estimates given in (3.5), namely that

$$V(x_0, 2^{j_{k-1}}2^{i_0}\sigma)^{-\frac{1}{2}-\frac{\eta-\epsilon}{\omega}} \leq (2C_{\mu})^{\frac{1}{2}+\frac{\eta-\epsilon}{\omega}} V(x_0, 2^{j_k}2^{i_0}\sigma)^{-\frac{1}{2}-\frac{\eta-\epsilon}{\omega}}$$

implies

$$\begin{aligned} \|m_k\|_{L^2} &\leq 2V(x_0, 2^{j_{k-1}}2^{i_0}\sigma)^{-\frac{1}{2}-\frac{\eta-\epsilon}{\omega}} \sigma^{\frac{1}{\alpha}\left(\frac{\omega+2\eta-2\epsilon}{\omega} \frac{p}{2-p}-1\right)} \\ (3.8) \quad &\leq 2 \cdot 2^{-\frac{1}{\omega\alpha}\left(\frac{\omega+2\eta-2\epsilon}{\omega} \frac{p}{2-p}-1\right)k} (2C_{\mu})^{\frac{1}{2}+\frac{\eta-\epsilon}{\omega}} C_{\mu}^{\frac{1}{\omega\alpha}\left(\frac{\omega+2\eta-2\epsilon}{\omega} \frac{p}{2-p}-1\right)} V(x_0, 2^{j_k}2^{i_0}\sigma)^{\left(\frac{1}{2}-\frac{1}{p}\right)}. \end{aligned}$$

Thus, $2^{-1} \cdot 2^{\frac{1}{\omega\alpha}\left(\frac{\omega+2\eta-2\epsilon}{\omega} \frac{p}{2-p}-1\right)k} (2C_{\mu})^{-\frac{1}{2}-\frac{\eta-\epsilon}{\omega}} C_{\mu}^{-\frac{1}{\omega\alpha}\left(\frac{\omega+2\eta-2\epsilon}{\omega} \frac{p}{2-p}-1\right)} m_k$ are $(p, 2)$ atoms.

Moreover, $\sum_k 2^{-\frac{1}{\omega\alpha}(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1)kp} < \infty$. As a consequence, $\sum_{k=0}^{\infty} m_k(x)$ gives the desired atomic decomposition and hence, by Theorem 1.1, belongs to $H^p(X)$ with the norm not larger than the constant C , which depends only on $p, \omega, \eta, \epsilon$ and C_μ .

It remains to show that $\sum_{k=0}^{\infty} \bar{m}_k \tilde{\chi}_{j_k}(x)$ also gives an atomic decomposition. To see this, let $N_{k'} = \sum_{k=k'}^{\infty} \bar{m}_k$. Note that $\sum_{k=0}^{\infty} \bar{m}_k = \int m(x) d\mu(x) = 0$. Summing up by parts implies that

$$\sum_{k=0}^{\infty} \bar{m}_k \tilde{\chi}_k(x) = \sum_{k'=0}^{\infty} (N_{k'} - N_{k'+1}) \tilde{\chi}_{k'}(x) = \sum_{k'=0}^{\infty} N_{k'+1} (\tilde{\chi}_{k'+1}(x) - \tilde{\chi}_{k'}(x)).$$

Observe that the support of $\tilde{\chi}_{k'+1}(x) - \tilde{\chi}_{k'}(x)$ lies within $B(x_0, 2^{k'+1}\sigma)$ and

$$\int_X (\tilde{\chi}_{k'+1}(x) - \tilde{\chi}_{k'}(x)) d\mu(x) = 0$$

since $\int \tilde{\chi}_{k'}(x) d\mu(x) = 1$ for all k' . And we also have

$$|\tilde{\chi}_{k'+1}(x) - \tilde{\chi}_{k'}(x)| \leq \frac{1}{\int \tilde{\chi}_{k'+1}(y) d\mu(y)} + \frac{1}{\int \tilde{\chi}_{k'}(y) d\mu(y)} \leq \frac{2}{\int \tilde{\chi}_{k'}(y) d\mu(y)} = \frac{2}{\mu(\tilde{\chi}_{k'})}.$$

Now applying (3.6), we obtain that

$$(3.9) \quad |\tilde{\chi}_{k'+1}(x) - \tilde{\chi}_{k'}(x)| \leq \frac{8C_\mu}{\mu(B(x_0, 2^{j_{k'+1}} 2^{i_0} \sigma))}.$$

Applying the Hölder inequality and the estimates in (3.8), we obtain that

$$\begin{aligned} |N_{k'+1}| &\leq \sum_{k=k'+1}^{\infty} \int |m(x) \tilde{\chi}_{j_k}(x)| d\mu(x) \\ &\leq C \sum_{k=k'+1}^{\infty} \left(\int_{\tilde{\chi}_{j_k}} |m(x)|^2 d\mu(x) \right)^{1/2} \mu(\tilde{\chi}_{j_k})^{1/2} \\ &\leq C \sum_{k=k'+1}^{\infty} 2 \cdot 2^{-\frac{1}{\omega\alpha}(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1)k} (2C_\mu)^{\frac{1}{2} + \frac{\eta-\epsilon}{\omega}} C_\mu^{\frac{1}{\omega\alpha}(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1)} \\ &\quad \times V(x_0, 2^{j_k} 2^{i_0} \sigma)^{(\frac{1}{2} - \frac{1}{p})} \mu(B(x_0, 2^{j_k} 2^{i_0} \sigma))^{1/2} \\ &\leq C 2 \cdot 2^{-\frac{1}{\omega\alpha}(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1)(k'+1)} (2C_\mu)^{\frac{1}{2} + \frac{\eta-\epsilon}{\omega}} C_\mu^{\frac{1}{\omega\alpha}(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1)} \\ &\quad \times \mu(B(x_0, 2^{j_{k'+1}} 2^{i_0} \sigma))^{1 - \frac{1}{p}}. \end{aligned}$$

The estimate above and the size estimate of $\tilde{\chi}_{k'+1}(x) - \tilde{\chi}_{k'}(x)$ in (3.9) imply

$$\begin{aligned} |N_{k'+1}(\tilde{\chi}_{k'+1}(x) - \tilde{\chi}_{k'}(x))| &\leq C 2 \cdot 2^{-\frac{1}{\omega\alpha}(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1)(k'+1)} (2C_\mu)^{\frac{1}{2} + \frac{\eta-\epsilon}{\omega}} C_\mu^{\frac{1}{\omega\alpha}(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1)} \\ &\quad \times \frac{6C_\mu}{\mu(B(x_0, 2^{j_{k'+1}} 2^{i_0} \sigma))} \mu(B(x_0, 2^{j_{k'+1}} 2^{i_0} \sigma))^{1 - \frac{1}{p}} \\ &\leq C 2^{-\frac{1}{\omega\alpha}(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1)(k'+1)} \mu(B(x_0, 2^{j_{k'+1}} 2^{i_0} \sigma))^{-\frac{1}{p}}. \end{aligned}$$

Therefore, we can rewrite $N_{k'+1}(\tilde{\chi}_{k'+1}(x) - \tilde{\chi}_{k'}(x))$ as

$$N_{k'+1}(\tilde{\chi}_{k'+1}(x) - \tilde{\chi}_{k'}(x)) = \alpha_{k'} \beta_{k'}(x),$$

where $\alpha_{k'} = C2^{-\frac{1}{\omega\alpha}(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1)(k'+1)}$ and $\beta_{k'}(x)$ are $(p, 2)$ atoms. Hence, by Theorem 1.1, $\sum_{k=0}^{\infty} \overline{m}_k \widetilde{\chi}_k(x)$ belongs to $H^p(X)$ with the $H^p(X)$ norm does not exceed C . The proof of Theorem 3.2 is concluded. \square

3.2. The Proof of Theorem 1.3.

Proof of Theorem 1.3. Let $\frac{\omega}{\omega+\eta} < p \leq 1$ and $\epsilon > 0$ such that $\frac{\omega}{\omega+\eta-\epsilon} < p \leq 1$. Suppose that T is a singular integral operator with kernel $K(x, y)$ satisfying the estimate (1.14) and (1.16), and T is bounded on $L^2(X)$. Note that $L^2(X) \cap H^p(X)$ is dense in $H^p(X)$ and if $f \in L^2(X) \cap H^p(X)$ then f has an atomic decomposition $f = \sum_j \lambda_j a_j$ where the series converges in both $L^2(X)$ and $H^p(X)$. Therefore, to show that T extends to be a bounded operator on $H^p(X)$, it suffices to verify that for each $(p, 2)$ -atom a , $m = T(a)$ is an $(p, 2, \epsilon)$ -molecule up to a multiplication of a constant C . Suppose that a is an $(p, 2)$ atom with the support $B(x_0, r)$. We write

$$\begin{aligned} & \left(\int_X m(x)^2 d\mu(x) \right) \left(\int_X m(x)^2 V(x_0, x)^{1+\frac{2\eta-2\epsilon}{\omega}} d\mu(x) \right)^{\left(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1\right)^{-1}} \\ & \leq \left(\int_X m(x)^2 d\mu(x) \right) \left(\int_{d(x_0, x) \leq 2r} m(x)^2 V(x_0, x)^{1+\frac{2\eta-2\epsilon}{\omega}} d\mu(x) \right)^{\left(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1\right)^{-1}} \\ & \quad + \left(\int_X m(x)^2 d\mu(x) \right) \left(\int_{d(x_0, x) > 2r} m(x)^2 V(x_0, x)^{1+\frac{2\eta-2\epsilon}{\omega}} d\mu(x) \right)^{\left(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1\right)^{-1}} \\ & := I + II. \end{aligned}$$

Note that, by the L^2 boundedness of T and the size condition on a , $\|m\|_{L^2}^2 \leq V(x_0, r)^{(1-\frac{2}{p})}$. As for I , applying the doubling property on μ implies that

$$\begin{aligned} I & \leq CV(x_0, r)^{(1-\frac{2}{p})} V(x_0, 2r)^{(1+\frac{2\eta-2\epsilon}{\omega})(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1)^{-1}} \left(\int m(x)^2 d\mu(x) \right)^{\left(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1\right)^{-1}} \\ & \leq CV(x_0, r)^{(1-\frac{p}{2})} V(x_0, r)^{(1+\frac{2\eta-2\epsilon}{\omega})(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1)^{-1}} V(x_0, r)^{(1-\frac{2}{p})(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1)^{-1}} \\ & \leq C. \end{aligned}$$

To estimate II , observe that if $d(x_0, x) > 2r$, by the cancellation condition on a , we have

$$\begin{aligned} |m(x)| &= \int [K(x, y) - K(x, x_0)] a(y) d\mu(y) \leq C \int \frac{1}{V(x_0, x)} \left(\frac{d(x_0, y)}{d(x_0, x)} \right)^\eta |a(y)| d\mu(y) \\ &\leq C \frac{1}{V(x_0, x)} \left(\frac{r}{d(x_0, x)} \right)^\eta \mu(B(x_0, r))^{1-\frac{1}{p}}. \end{aligned}$$

This together with the doubling property on μ gives

$$\begin{aligned} II & \leq C\mu(B(x_0, r))^{(1-\frac{2}{p})} \left(\int_{d(x_0, x) > 2r} m(x)^2 V(x_0, x)^{1+\frac{2\eta-2\epsilon}{\omega}} d\mu(x) \right)^{\left(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1\right)^{-1}} \\ & \leq C\mu(B(x_0, r))^{(1-\frac{2}{p})} \\ & \quad \times \left(\int_{d(x_0, x) > 2r} \frac{1}{V(x_0, x)^2} \left(\frac{r}{d(x_0, x)} \right)^{2\eta} V(x_0, x)^{1+\frac{2\eta-2\epsilon}{\omega}} \mu(B(x_0, r))^{2-\frac{2}{p}} d\mu(x) \right)^{\left(\frac{\omega+2\eta-2\epsilon}{\omega}\frac{p}{2-p}-1\right)^{-1}} \end{aligned}$$

$$\begin{aligned}
&\leq C\mu(B(x_0, r))^{(1-\frac{2}{p})} \\
&\quad \times \left(r^{2\eta} \int_{d(x_0, x) > 2r} \frac{V(x_0, x)^{\frac{2\eta-2\epsilon}{\omega}}}{V(x_0, x)} \left(\frac{1}{d(x, x_0)} \right)^{2\eta} \mu(B(x_0, r))^{2-\frac{2}{p}} d\mu(x) \right)^{(\frac{\omega+2\eta-2\epsilon}{\omega} \frac{p}{2-p} - 1)^{-1}} \\
&\leq C\mu(B(x_0, r))^{(1-\frac{2}{p})} \left(r^{2\eta} \sum_{j \geq 0} \int_{2^{j+1}r \leq d(x, x_0) < 2^{j+2}r} \frac{V(x_0, 2^{j+2}r)^{\frac{2\eta-2\epsilon}{\omega}}}{V(x_0, 2^{j+1}r)} \right. \\
&\quad \times \left. \left(\frac{1}{2^{j+1}r} \right)^{2\eta} \mu(B(x_0, r))^{2-\frac{2}{p}} d\mu(x) \right)^{(\frac{\omega+2\eta-2\epsilon}{\omega} \frac{p}{2-p} - 1)^{-1}} \\
&\leq C\mu(B(x_0, r))^{(1-\frac{2}{p})} \left(\mu(B(x_0, r))^{\frac{2\eta-2\epsilon}{\omega} + 2 - \frac{2}{p}} \right)^{(\frac{\omega+2\eta-2\epsilon}{\omega} \frac{p}{2-p} - 1)^{-1}} \\
&\leq C.
\end{aligned}$$

Finally, by the fact that $T^*(1) = 0$, we obtain that $\int m(x) d\mu(x) = \int T(a)(x) d\mu(x) = 0$ and hence m is the multiple of an $(p, 2, \epsilon)$ molecule. The proof of the sufficient implication of Theorem 1.3 then follows from Theorem 3.2.

We now prove that if T is bounded on $L^2(X)$ and on $H^p(X)$ then $\int T(f)(x) d\mu(x) = 0$ for $f \in L^2(X) \cap H^p(X)$. As mentioned in section 1, this follows from the following general result.

Proposition 3.3. *If $f \in L^2(X) \cap H^p(X)$, $\frac{\omega}{\omega+\eta} < p \leq 1$, then there exists a constant C independent of the $L^2(X)$ norm of f such that*

$$(3.10) \quad \|f\|_p \leq C \|f\|_{H^p}.$$

As a consequence of this proposition, if $f \in L^2(X) \cap H^p(X)$, then $f \in L^1(X)$ and $\int f(x) d\mu(x) = 0$. Indeed, if $f \in L^2(X) \cap H^p(X)$, by Proposition 3.3, then $f \in L^p(X) \cap L^2(X)$. Hence, by interpolation, $f \in L^1(X)$. To see the integral of f is zero, applying the wavelet expansion as in Theorem 1.1, $f(x) = \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{Y}^k} \langle f, \psi_\alpha^k \rangle \psi_\alpha^k(x)$ where the series converges in both $L^2(X)$ and $H^p(X)$. Let $E_n(k, \alpha)$ be a finite set of $k \in \mathbb{Z}$ and $\alpha \in \mathcal{Y}^k$ and $E_n(k, \alpha)$ tends to the whole set $\{(k, \alpha) : k \in \mathbb{Z}, \alpha \in \mathcal{Y}^k\}$. Therefore, $\sum_{E_n^c(k, \alpha)} \langle f, \psi_\alpha^k \rangle \psi_\alpha^k(x)$ converges to zero as n tends to infinity in both $L^2(X)$ and $H^p(X)$. We obtain that

$$\begin{aligned}
\left| \int f(x) d\mu(x) \right| &\leq \left| \int \sum_{E_n(k, \alpha)} \langle f, \psi_\alpha^k \rangle \psi_\alpha^k(x) d\mu(x) \right| + \left| \int \sum_{E_n^c(k, \alpha)} \langle f, \psi_\alpha^k \rangle \psi_\alpha^k(x) d\mu(x) \right| \\
&\leq \left| \int \sum_{E_n^c(k, \alpha)} \langle f, \psi_\alpha^k \rangle \psi_\alpha^k(x) d\mu(x) \right| \\
&\leq C \left\| \sum_{E_n^c(k, \alpha)} \langle f, \psi_\alpha^k \rangle \psi_\alpha^k(x) \right\|_{H^p} + C \left\| \sum_{E_n^c(k, \alpha)} \langle f, \psi_\alpha^k \rangle \psi_\alpha^k(x) \right\|_2,
\end{aligned}$$

where the first inequality follows from the fact that $\int \sum_{E_n(k, \alpha)} \langle f, \psi_\alpha^k \rangle \psi_\alpha^k(x) d\mu(x) = 0$ since the cancellation property on the wavelet $\psi_\alpha^k(x)$.

Letting n tend to infinity gives the desired result.

Now assuming Proposition 3.3 for the moment, if T is bounded on both $L^2(X)$ and $H^p(X)$ and $f \in L^2(X) \cap H^p(X)$, then $Tf \in L^2(X) \cap H^p(X)$ and hence $\int Tf(x)d\mu(x) = 0$. The necessary implication of Theorem 1.3 is concluded.

It remains to show Proposition 3.3. The key idea of the proof is to apply the method of atomic decomposition for subspace $L^2(X) \cap H^p(X)$. More precisely, if $f \in L^2(X) \cap H^p(X)$, as in the proof of Theorem 1.1, we set

$$\Omega_l = \left\{ x \in X : \tilde{S}(f)(x) > 2^l \right\},$$

$$B_l = \left\{ Q_\alpha^k : \mu(Q_\alpha^k \cap \Omega_l) > \frac{1}{2}\mu(Q_\alpha^k) \text{ and } \mu(Q_\alpha^k \cap \Omega_{l+1}) \leq \frac{1}{2}\mu(Q_\alpha^k) \right\}$$

and

$$\tilde{\Omega}_l = \{x \in X : M(\chi_{\Omega_l})(x) > 1/2\},$$

where M is the Hardy–Littlewood maximal function on X and hence, $\mu(\tilde{\Omega}_l) \leq C\mu(\Omega_l)$.

Applying Proposition 2.5, we write

$$f(x) = \sum_l \sum_{Q_\alpha^{k+N} \in B_l} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}),$$

where the series converges in both $L^2(X)$ and $H^p(X)$. Thus, for $\frac{\omega\omega+\eta}{<}p \leq 1$,

$$\|f(x)\|_p^p \leq \sum_l \left\| \sum_{Q_\alpha^{k+N} \in B_l} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}) \right\|_p^p.$$

Note that if $Q_\alpha^{k+N} \in B_l$ then $Q_\alpha^{k+N} \subset \tilde{\Omega}_l$. Therefore, $\sum_{Q_\alpha^{k+N} \in B_l} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N})$ is supported in $\tilde{\Omega}_l$. Applying Hölder inequality implies that

$$\|f(x)\|_p^p \leq \sum_l \mu(\tilde{\Omega}_l)^{1-\frac{p}{2}} \left\| \sum_{Q_\alpha^{k+N} \in B_l} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}) \right\|_2^p.$$

As in the proof of Theorem 1.1, we have

$$\left\| \sum_{Q_\alpha^{k+N} \in B_l} \mu(Q_\alpha^{k+N}) D_k(x, x_\alpha^{k+N}) \tilde{D}_k(g)(x_\alpha^{k+N}) \right\|_2 \leq C 2^l \mu(\tilde{\Omega}_l)^{\frac{1}{2}},$$

which gives

$$\|f(x)\|_p^p \leq C \sum_l 2^{lp} \mu(\Omega_l)^p \leq C \|\tilde{S}(f)\|_p^p \leq C \|f\|_{H^p}^p$$

since $\mu(\tilde{\Omega}_l) \leq C\mu(\Omega_l)$.

The proof of Theorem 1.3 is concluded. \square

4. CRITERION OF THE BOUNDEDNESS FOR SINGULAR INTEGRALS ON CARLESON MEASURE AND CAMPANATO SPACES

As mentioned in Section 1, we first prove the following the weak density argument.

Lemma 4.1. *Suppose that $\frac{\omega}{\omega+\eta} < p \leq 1$. Then $L^2(X) \cap \text{CMO}^p(X)$ is dense in $\text{CMO}^p(X)$ in the sense of the weak topology $(H^p(X), \text{CMO}^p(X))$. Moreover precisely, for each $f \in \text{CMO}^p(X)$ there exists a sequence $\{f_n\} \in L^2(X) \cap \text{CMO}^p(X)$ such that $\|f_n\|_{\text{CMO}^p} \leq \|f\|_{\text{CMO}^p}$ and moreover,*

$$\lim_{n \rightarrow \infty} \langle f_n, g \rangle = \langle f, g \rangle$$

for all $g \in H^p(X)$.

Proof. Let f be in $\text{CMO}^p(X)$. By wavelet expansion, $f(x) = \sum_{k,\alpha} \langle \psi_\alpha^k, f \rangle \psi_\alpha^k(x)$, we set

$$f_n(x) := \sum_{|k| \leq n} \sum_{Q_\alpha^k \subset B_n} \langle \psi_\alpha^k, f \rangle \psi_\alpha^k(x),$$

where $B_n = \{x : d(x_0, x) \leq n\}$ and $x_0 \in X$ is fixed. It is easy to see that $f_n \in L^2(X)$. To see that $f_n \in \text{CMO}^p(X)$, for any quasi dyadic cube $P \subset X$ in the sense of Auscher and Hytönen,

$$\begin{aligned} & \sup_P \frac{1}{\mu(P)^{\frac{2}{p}-1}} \sum_{k' \in \mathbb{Z}, \alpha' \in \mathcal{Y}^{k'}, Q_{\alpha'}^{k'} \subset P} |\langle \psi_{\alpha'}^{k'}, f_n \rangle|^2 \\ &= \sup_P \frac{1}{\mu(P)^{\frac{2}{p}-1}} \sum_{|k| \leq n, Q_{\alpha'}^{k'} \subset P \cap B_n} |\langle \psi_{\alpha'}^{k'}, f \rangle| \leq \|f\|_{\text{CMO}^p}^2, \end{aligned}$$

which implies that $\|f_n\|_{\text{CMO}^p} \leq \|f\|_{\text{CMO}^p}$.

We now verify that f_n converges to f in the weak topology $(H^p(X), \text{CMO}^p(X))$. To do this, for any $h \in \mathring{G}(\beta, \gamma)$, by the wavelet expansion,

$$\langle f - f_n, h \rangle = \langle f, \sum_{|k| > n, \text{or } Q_\alpha^k \not\subset B_n} \psi_\alpha^k \langle \psi_\alpha^k, h \rangle \rangle.$$

Note that h is in both $H^p(X)$ and $\text{CMO}^p(X)$ and moreover, $\sum_{|k| > n, \text{or } Q_\alpha^k \not\subset B_n} \psi_\alpha^k(x) \langle \psi_\alpha^k, h \rangle$ tends to zero as n tends to infinity in the $H^p(X)$ norm. Therefore, by the duality argument in Theorem C, $\langle f - f_n, h \rangle$ tends to 0 as n tends to infinity. Further note that $\mathring{G}(\beta, \gamma)$ is dense in $H^p(X)$. Thus, for each $g \in H^p(X)$ and for any $\varepsilon > 0$, there exists a function $h \in \mathring{G}(\beta, \gamma)$ such that $\|g - h\|_{H^p(X)} < \varepsilon$. Now by the duality and the fact that $\|f_n\|_{\text{CMO}^p(X)} \leq \|f\|_{\text{CMO}^p(X)}$, we have

$$\begin{aligned} |\langle f - f_n, g \rangle| &\leq |\langle f - f_n, g - h \rangle| + |\langle f - f_n, h \rangle| \\ &\leq \|f - f_n\|_{\text{CMO}^p} \|g - h\|_{H^p} + |\langle f - f_n, h \rangle| \\ &\leq 2\varepsilon \|f\|_{\text{CMO}^p} + |\langle f - f_n, h \rangle|, \end{aligned}$$

which together with the fact that $|\langle f - f_n, h \rangle|$ tends to zero as n tends to infinity implies that $\lim_{n \rightarrow \infty} \langle f - f_n, g \rangle = 0$. The proof of Lemma 4.1 is completed. \square

We now return to the proof of Theorem 1.4. Suppose that T is a singular integral with the kernel $K(x, y)$ satisfying the estimates (1.14) and (1.15), and T is bounded on $L^2(X)$. To show the boundedness of T on $\text{CMO}^p(X)$, we first define the action of T on $\text{CMO}^p(X)$. To do this, given $f \in \text{CMO}^p(X)$, by Lemma 4.1, there exists a sequence $\{f_n\}$ converges to f in the weak topology $(H^p(X), \text{CMO}^p(X))$. We observe that for all $g \in H^p(X)$, $\langle T f_n, g \rangle$ has the limit as n tends to infinity. This is because that T^* satisfies all conditions of Theorem 1.3 and hence T^* is bounded on $H^p(X)$. Thus, if g is in the subspace $L^2(X) \cap H^p(X)$, then $T^*(g) \in L^2(X) \cap H^p(X)$, by the duality in Theorem C,

$$|\langle T f_n, g \rangle - \langle T f_m, g \rangle| = |\langle f_n - f_m, T^* g \rangle|,$$

which tends to zero as n, m tend to infinity and hence $\langle T f_n, g \rangle$ has the limit as n tends to infinity.

Applying the density argument gives the observation. Note that the limit of $\langle T f_n, g \rangle$ as n tends to infinity does not depend on the sequence $\{f_n\}$ satisfying Lemma 4.1 and thus, we can define

$$\langle T f, g \rangle = \lim_{n \rightarrow \infty} \langle T f_n, g \rangle$$

for all $g \in H^p(X)$.

We claim that there exists a constant C such that

$$\|T f\|_{\text{CMO}^p} \leq C \|f\|_{\text{CMO}^p}$$

for all $f \in L^2(X) \cap \text{CMO}^p(X)$.

Assuming the claim for the moment, by the definition of $T f$ for $f \in \text{CMO}^p(X)$, we have $\langle \psi_\alpha^k, T f \rangle = \lim_{n \rightarrow \infty} \langle \psi_\alpha^k, T f_n \rangle$ where f_n is the sequence given by Lemma 4.1. This together with the claim implies that

$$\begin{aligned} \|T f\|_{\text{CMO}^p} &\leq \liminf_{n \rightarrow \infty} \|T f_n\|_{\text{CMO}^p} \\ &\leq C \liminf_{n \rightarrow \infty} \|f_n\|_{\text{CMO}^p} \leq C \|f\|_{\text{CMO}^p}, \end{aligned}$$

which completes the proof of the sufficiency of Theorem 1.4.

We return to show the claim. The basic idea is to apply the duality in Theorem C. To be more precise, by Theorem C, for $f \in L^2(X) \cap \text{CMO}^p(X)$ and $g \in L^2(X) \cap H^p(X)$,

$$|\langle T f, g \rangle| = |\langle f, T^* g \rangle| \leq C \|f\|_{\text{CMO}^p} \|T^* g\|_{H^p} \leq C \|f\|_{\text{CMO}^p} \|g\|_{H^p}.$$

This implies that for each $f \in L^2(X) \cap \text{CMO}^p(X)$, $\mathcal{L}_f(g) = \langle T f, g \rangle$ defines a continuous linear functional on $L^2(X) \cap H^p(X)$. Note that $L^2(X) \cap H^p(X)$ is dense in $H^p(X)$. Thus, $\mathcal{L}_f(g) = \langle T(f), g \rangle$ extends to a linear functional on $H^p(X)$ and the norm of this linear

functional is dominated by $C\|f\|_{\text{CMO}^p(X)}$. By Theorem C, there exists $h \in \text{CMO}^p(X)$ with $\|h\|_{\text{CMO}^p} \leq C\|f\|_{\text{CMO}^p}$ such that

$$\langle Tf, g \rangle = \mathcal{L}_f(g) = \langle h, g \rangle$$

for $g \in L^2(X) \cap H^p(X)$.

Particularly, note that all $\psi_\alpha^k(x) \in L^2(X) \cap H^p(X)$. Therefore,

$$\langle Tf, \psi_\alpha^k \rangle = \langle h, \psi_\alpha^k \rangle,$$

which, by the definition of space $\text{CMO}^p(X)$, we obtain that

$$\|Tf\|_{\text{CMO}^p} = \|h\|_{\text{CMO}^p} \leq C\|f\|_{\text{CMO}^p}.$$

The proof of the claim is concluded.

The necessary condition in Theorem 1.4 follows directly from the boundedness of T on $\text{CMO}^p(X)$ and the fact that the function 1 has the norm zero in $\text{CMO}^p(X)$. The proof of Theorem 1.4 is concluded.

As a consequence of Theorem 1.4, we have the following

Corollary 4.2. *If T satisfies the same condition as in Theorem 1.4, then T is bounded on Campanato space $\mathcal{C}_{\frac{1}{p}-1}(X)$ if and only if $T(1) = 0$.*

The proof of this corollary follows directly from Theorem 1.4 and the following

Proposition 4.3. $\text{CMO}^p(X) = \mathcal{C}_{\frac{1}{p}-1}(X)$, $\frac{\omega}{\omega+\eta} < p \leq 1$ with equivalent norms.

Proof. By Lemma 2.6, we only need to show that there exists a constant C such that for $\frac{\omega}{\omega+\eta} < p \leq 1$,

$$\|f\|_{\mathcal{C}_{\frac{1}{p}-1}} \leq C\|f\|_{\text{CMO}^p}.$$

The basic idea to verify the above estimate is to apply the duality argument in Theorem C. To this end, for given $f \in \text{CMO}^p(X)$, $\frac{\omega}{\omega+\eta} < p \leq 1$, by Theorem C, we can define a linear functional on $H^p(X)$ by $\mathcal{L}_f(g) = \langle f, g \rangle$ for $g \in H^p(X)$. Now fix a quasi-dyadic ball Q and let L_Q^2 denote the space of all square integrable functions supported on Q . Let $L_{Q,0}^2$ denote its closed subspaces of functions with integral zero. Note that each $g \in L_{Q,0}^2$ is a multiple of an $(p, 2)$ atom for $H^p(X)$ and that $\|g\|_{H^p(X)} \leq \mu(Q)^{\frac{1}{p}-\frac{1}{2}}\|g\|_{L^2}$. Therefore, this linear functional \mathcal{L}_f on $H^p(X)$ can extend to a linear functional on $L_{Q,0}^2$ with norm at most $C\mu(Q)^{\frac{1}{p}-\frac{1}{2}}\|f\|_{\text{CMO}^p}$. By the Riesz representation theorem for Hilbert spaces $L_{Q,0}^2$, there exists an element $F^Q \in L_{Q,0}^2$ such that

$$\mathcal{L}_f(g) = \langle f, g \rangle = \int_Q F^Q(x)g(x)d\mu(x), \quad \text{if } g \in L_{Q,0}^2,$$

with

$$\|F^Q\|_{L_Q^2} = \left\{ \int_Q |F^Q(x)|^2 d\mu(x) \right\}^{\frac{1}{2}} \leq C\mu(Q)^{\frac{1}{p}-\frac{1}{2}}\|f\|_{\text{CMO}^p}.$$

Thus f must be a square integrable function on Q and for each quasi ball Q , we have such a function F^Q such that on each ball Q , f differ from F^Q by a constant. This implies that f

is a locally square integrable function on X and on each quasi ball Q there exists a constant c_Q such that $f = F^Q + c_Q$. Observe that

$$\begin{aligned} \left\{ \frac{1}{\mu(Q)^{\frac{2}{p}-1}} \int_Q |f(x) - c_Q|^2 d\mu(x) \right\}^{1/2} &= \left\{ \frac{1}{\mu(Q)^{\frac{2}{p}-1}} \int_Q |F^Q(x)|^2 d\mu(x) \right\}^{1/2} \\ &\leq C \frac{1}{\mu(Q)^{\frac{1}{p}-\frac{1}{2}}} \mu(Q)^{\frac{1}{p}-\frac{1}{2}} \|f\|_{\text{CMO}^p} \\ &= C \|f\|_{\text{CMO}^p}. \end{aligned}$$

Note that on each quasi ball Q and for any constant c , $\int_Q |f(x) - c|^2 d\mu(x) \leq C \int_Q |f(x) - c_Q|^2 d\mu(x)$. The above estimate yields that

$$\left\{ \frac{1}{\mu(Q)^{\frac{2}{p}-1}} \int_Q |f(x) - f_Q|^2 d\mu(x) \right\}^{1/2} \leq C \|f\|_{\text{CMO}^p},$$

which implies that $f \in \mathcal{L}_{\frac{1}{p}-1}(X)$ and

$$\|f\|_{\mathcal{L}_{\frac{1}{p}-1}(X)} \leq C \|f\|_{\text{CMO}^p}.$$

□

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